

Calculating Gluino-Condensate Prepotential

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Abstract

We discuss the derivation of the CIV-DV prepotential for arbitrary power $n + 1$ of the original superpotential in the $N = 1$ SUSY YM theory (for arbitrary number n of cuts in the solution of the planar matrix model in the Dijkgraaf-Vafa interpretation). The goal is to hunt for structures, not so much for exact formulas, which are necessarily complicated, before the right language is found to represent them. Some entities, reminiscent of representation theory, clearly emerge, but a lot of work remains to be done to identify the relevant ones. As a practical application, we obtain a cubic (first non-perturbative) contribution to the prepotential for any n .

After the recent papers of R.Dijkgraaf and C.Vafa [1] the calculation of gluino-condensate prepotential [2] attracted wide attention, and many results are already obtained [3]-[6] for various examples, starting both from the planar matrix models solutions [7] and from the Seiberg-Witten (SW) theory [8, 9, 5]. However, not enough attention is yet devoted to the general structure of the *multi-cut* prepotentials (with any number n of cuts, i.e. for any power $n + 1$ of the superpotential, i.e. for any genus $n - 1$ of the spectral curve). At the same time, the precise understanding of this structure is of crucial importance for establishing relations between the Dijkgraaf-Vafa (DV) theory and other branches of mathematical physics and string theory, such as the theory of effective actions in the spirit of [10], representation theory, finite- N matrix models [11], KP/Toda-like τ -functions [12] and their generalizations [13], instanton calculus *a la* refs.[14, 15, 4], WDVV equations [16, 6] etc.

1 The plan of calculation

We want to evaluate the set of integrals

$$\begin{aligned} \frac{1}{g_{n+1}} S_i &= 2 \int_{\gamma_i - \rho_i}^{\gamma_i + \rho_i} dS_{DV}, \\ \frac{1}{g_{n+1}} \Pi_i &= 2 \int_{\gamma_i + \rho_i}^{\Lambda} dS_{DV} \end{aligned} \tag{1}$$

($i = 1, \dots, n$), which are the periods of the meromorphic (with poles at infinities) differential

$$dS_{DV} = y(x)dx \quad (2)$$

on the hyperelliptic complex curve (Riemann surface) of genus $n - 1$ with quadratic branching points at $\gamma_i \pm \rho_i$:

$$y^2(x) = \prod_{i=1}^n (x - \gamma_i - \rho_i)(x - \gamma_i + \rho_i) = \prod_{i=1}^n ((x - \gamma_i)^2 - \rho_i^2) \quad (3)$$

The periods Π_i are to be further interpreted as the S_i -derivatives of the prepotential,

$$\Pi_i = \left. \frac{\partial \mathcal{F}}{\partial S_i} \right|_{\text{constant } \alpha's} \quad (4)$$

evaluated at constant values of parameters α_i , which are defined to be the roots of a polynomial

$$P_n(x) = \prod_{i=1}^n (x - \alpha_i) \quad (5)$$

of degree n , such that the curve (3) is also

$$y^2(x) = P_n^2(x) + f_{n-1}(x), \quad (6)$$

where the polynomial

$$f_{n-1}(x) = \sum_{i=1}^n \tilde{S}_i \prod_{j \neq i}^n (x - \alpha_j) = P_n(x) \sum_{i=1}^n \frac{\tilde{S}_i}{x - \alpha_i} \quad (7)$$

is of degree $n - 1$ only. According to the general arguments of the SW theory [8, 9, 5], such system of equations is consistent, i.e.

$$\left. \frac{\partial \Pi_i}{\partial S_j} \right|_{\text{constant } \alpha's} = \left. \frac{\partial \Pi_j}{\partial S_i} \right|_{\text{constant } \alpha's} \quad (8)$$

because the variation

$$\delta dS_{DV} = \sum_{i=1}^n \delta \tilde{S}_i dv_i(x) + O(\delta \alpha) \quad (9)$$

for $\delta \alpha_i = 0$ is a linear combination of (almost) holomorphic differentials

$$dv_i(x) = \frac{\prod_{j \neq i}^n (x - \alpha_j)}{y(x)} dx \quad (10)$$

on the curve (3). In fact $dv_i(x)$ possess simple poles at $x = \infty_{\pm}$, but this does not spoil the symmetric property of the period matrix (8) in the limit $\Lambda \rightarrow \infty$ (deviations from (8) are of the order $O(\alpha/\Lambda)$).

This means that the equations can be integrated and provide a function, the CIV-DV prepotential $\mathcal{F}(S|\alpha)$, which has the form of

$$\begin{aligned}
2\pi i \mathcal{F}(S|\alpha) = & 4\pi i g_{n+1} \left(W_{n+1}(\Lambda) \sum_i S_i - \sum_i W_{n+1}(\alpha_i) S_i \right) - \left(\sum_i S_i \right)^2 \log \Lambda + \\
& + \frac{1}{2} \sum_{i=1}^n S_i^2 \left(\log \frac{S_i}{4} - \frac{3}{2} \right) - \frac{1}{2} \sum_{i < j}^n (S_i^2 - 4S_i S_j + S_j^2) \log \alpha_{ij} + \sum_{k=1}^{\infty} \frac{1}{(i\pi g_{n+1})^k} \mathcal{F}_{k+2}(S|\alpha),
\end{aligned} \tag{11}$$

where $\mathcal{F}_{k+2}(S|\alpha)$ are polynomial of degree $k+2$ in S 's with α -dependent coefficients, $\alpha_{ij} = \alpha_i - \alpha_j$ and $W'_{n+1}(x) = P_n(x)$.

The problem of calculating $\mathcal{F}_{k+2}(S|\alpha)$ is technically involved.¹ Its solution includes 4 steps:²

- 1) Evaluation of integrals (1);
- 2) Inversion of formulas for $S_i(\rho, \gamma)$, providing expressions for $\rho(S, \gamma)$, and substitution of those into the Λ_i -independent ($\Lambda_i = \Lambda - \gamma_i$) pieces of integrals Π_i (the Λ_i -dependent contributions are better left expressed through γ and ρ at this stage);
- 3) Expression of γ 's through α 's (and ρ 's) by comparison of (3) and (6);
- 4) Substitution of the resulting expressions into those for Π 's, integrating over S 's and obtaining the desired expansion for the prepotential $\mathcal{F}(S|\alpha)$.

In what follows we present some results which allow us to reveal some of the structures; they emerge at every step of our procedure. There is of course, substantial overlap with the previous considerations in [1]-[6].

A practical outcome of our calculation is an expression for the first non-perturbative term in (11):³

$$\begin{aligned}
\mathcal{F}_3(S|\alpha) = & \sum_{i=1}^n u_i(\alpha) S_i^3 + \sum_{i \neq j}^n u_{i;j}(\alpha) S_i^2 S_j + \sum_{i < j < k}^n u_{ijk}(\alpha) S_i S_j S_k, \\
u_i(\alpha) = & \frac{1}{6} \left(- \sum_{j \neq i} \frac{1}{\alpha_{ij}^2 \Delta_j} + \frac{1}{4\Delta_i} \sum_{\substack{j < k \\ j, k \neq i}} \frac{1}{\alpha_{ij} \alpha_{ik}} \right), \\
u_{i;j}(\alpha) = & \frac{1}{4} \left(- \frac{3}{\alpha_{ij}^2 \Delta_i} + \frac{2}{\alpha_{ij}^2 \Delta_j} - \frac{2}{\alpha_{ij} \Delta_i} \sum_{k \neq i, j} \frac{1}{\alpha_{ik}} \right), \\
u_{ijk}(\alpha) = & \frac{1}{\alpha_{ij} \alpha_{ik} \Delta_i} + \frac{1}{\alpha_{ji} \alpha_{jk} \Delta_j} + \frac{1}{\alpha_{ki} \alpha_{kj} \Delta_k}, \\
\Delta_i = & W''_{n+1}(\alpha_i) = \prod_{j \neq i}^n \alpha_{ij}
\end{aligned} \tag{12}$$

In what follows, we begin with the step 1): evaluation of the integrals (1) in terms of γ and ρ . First, in s.2 we analyze the expressions for these integrals through symmetric functions of γ_{ij} (to be

¹An advantageous approach to it is provided by the DV interpretation of the same quantity in terms the planar large- N limit of matrix models, which allows to apply functional-integral techniques, in particular, the usual QFT perturbation theory (diagram expansion).

²Note some technical change as compared to s.5.3 of ref.[5]. Such minor modifications provide substantial technical simplifications.

³To match the overall coefficient to the one, frequently met in the literature, one should rescale S : our $S/4$ looks equal to S in ref.[2].

called $r(\gamma)$, $e(\gamma)$ and $\varepsilon(\gamma)$), certain elementary integrals (to be called J_m , not to be mixed with Bessel functions!) and peculiar infinite-order differential operator (to be called $\hat{D}(\rho, \gamma)$, this one actually *is* a Bessel function of $\rho^2 \partial_\gamma^2$). The main result of s.2 is eq.(35) and/or eq.(45), which may look more complicated, but is more useful for practical calculations. Eqs.(35) and (45) are valid even for *indefinite* integrals, expressions for concrete integration limits, explicit in (1) are obtained by substitution from (52) and (53). Then, in s.3 we concentrate on the first terms of expansion in powers of ρ^2 , relevant for evaluation of \mathcal{F}_3 : this means that we need ρ^0 , ρ^2 and ρ^4 . Then only a few types of symmetric functions contribute, and the situation can be drastically simplified. In the same s.3 we perform step 2), in application to the Λ_i -independent terms in Π_i , and this leads to new drastic simplifications. The Λ_i -dependent contributions to Π_i also get together into a simple expression, $W_{n+1}(\Lambda) - W_{n+1}(\gamma_i)$, but to see this one needs to rely upon the " γ vs α sum rules" (100) from Appendix A. The main result of s.3 consists the of simply-structured formulas (79), (80) for Π_i . Eq.(79) contains explicit γ -dependence (instead of being entirely α -dependent) in just two places: as an argument in $W_{n+1}(\gamma_i)$ and through $\log \gamma_{ij}$. Expression for γ through S and α , required at the next step 3) are derived in Appendix A, the main result of which is in eq.(110), although the sum rules (100) can sometime be directly applied as well, for example, to handle the Λ_i -dependent terms in Π_i . In s.4 we finally combine the results of s.3 and Appendix A to obtain the final formulas (11) and (12). Appendix B contains sample calculations for $n = 2$ and $n = 3$, which can be used for illustrative purposes.

2 Periods through cuts

2.1 Notation and basic integrals

For the purpose of evaluation of S_i and Π_i with given i , we expand dS_{DV} in powers of ρ_j with all $j \neq i$, keeping the ρ_i dependence exact in all terms of the expansion. This is needed because the integrals in question are not analytic in ρ_i and are not evaluated by expansion of the *integrands* in ρ_i .

Namely, we express $dS_{DV}(x)/dx = \sqrt{\mathcal{P}_{2n}(x)}$ through γ 's and ρ 's in the following way:

$$\frac{dS_{DV}(x)}{dx} = \sqrt{\mathcal{P}_{2n}(x)} = \sqrt{(x - \gamma_i)^2 - \rho_i^2} \prod_{j \neq i}^n \sqrt{(x - \gamma_j)^2 - \rho_j^2} \quad (13)$$

Since we are further going to integrate over x , we can shift the integration variable $x \rightarrow x_i + \gamma_i$, $\gamma_{ij} = \gamma_i - \gamma_j$ (and remember later to appropriately adjust the integration domains):

$$\begin{aligned} \int_i dS_{DV}(x) &= \int dx \sqrt{x^2 - \rho_i^2} \prod_{j \neq i}^n \sqrt{(x + \gamma_{ij})^2 - \rho_j^2} = \\ &= \int \sqrt{x^2 - \rho_i^2} \prod_{j \neq i}^n \left\{ \sum_{m_j=0}^{\infty} c_{m_j} \rho_j^{2m_j} (x + \gamma_{ij})^{1-2m_j} \right\} dx = \\ &= \sum_{\{m_1, \dots, m_n\}} \left(\prod_{j \neq i} c_{m_j} \rho_j^{2m_j} \right) I_{m_1, \dots, m_n}^{[i]}(\rho_i; \gamma_i) \end{aligned} \quad (14)$$

Here the coefficients⁴ $c_m = \frac{\Gamma(m-1/2)}{m! \Gamma(-1/2)}$ and integrals

⁴Somewhat amusingly, just the same c_m (or what is the same, square root expansions) appear in many places, which seem unrelated (though all can be traced back to the fact that we deal with a hyperelliptic curve with quadratic

$$I_{m_1, \dots, m_n}^{[i]}(\rho_i; \gamma_{i\cdot}) = \int \sqrt{x^2 - \rho_i^2} \prod_{j \neq i}^n (x + \gamma_{ij})^{1-2m_j} dx. \quad (15)$$

(Note that there are just $n - 1$ indices m_1, \dots, m_n – there is nothing standing at the i -th position. Also by $\gamma_{i\cdot}$, with a dot at the place of the second index, we denote the set $\{\gamma_{ij}\}$ with all possible values of $j \in \epsilon$, see below.)

Whenever some $m_j \geq 1$, one can use the recurrent formula,

$$I_{\dots m_j \dots}^{[i]}(\rho_i; \gamma_{i\cdot}) = \frac{1}{(2m_j - 2)!} \left(\frac{\partial}{\partial \gamma_{ij}} \right)^{2m_j - 2} I_{\dots 1_j \dots}^{[i]}(\rho_i; \gamma_{i\cdot}) \quad (16)$$

Having this formula in mind, it is reasonable to introduce more notation.

First, let us split the set of $n - 1$ numbers $N_n^{[i]} = \{1, 2, \dots, \check{i}, \dots, n\}$ (i is omitted) into two non-intersecting sets: ϵ , consisting of all k with $m_k = 0$ and $\bar{\epsilon}$, including all k with $m_k \geq 1$,

$$N_n^{[i]} = \epsilon \cup \bar{\epsilon}, \quad \epsilon \cap \bar{\epsilon} = \emptyset, \quad (17)$$

$$\epsilon \equiv \left\{ k \in N_n^{[i]} : m_k = 0 \right\}, \quad \bar{\epsilon} \equiv \left\{ k \in N_n^{[i]} : m_k > 0 \right\}$$

The numbers of elements in ϵ and $\bar{\epsilon}$ will be denoted by $d(\epsilon)$ and $d(\bar{\epsilon})$. Somewhat later we will also need

$$\epsilon_{j_1 \dots j_l} \equiv \epsilon / \{j_1, \dots, j_l\} = \left\{ k \in N_n^{[i]} : m_k = 0, k \neq j_1, \dots, j_l \right\} \quad (18)$$

(assuming that $j_1, \dots, j_l \in \epsilon$ and are all different), and similarly

$$\bar{\epsilon}_{j_1 \dots j_l} \equiv \bar{\epsilon} / \{j_1, \dots, j_l\} = \left\{ k \in N_n^{[i]} : m_k > 0, k \neq j_1, \dots, j_l \right\} \quad (19)$$

Second, we denote by $\sum_{\epsilon, \bar{\epsilon}}$ the sum over all the splittings of $N_n^{[i]}$ into two non-intersecting subspaces. The sum contains 2^{n-1} terms (the number of subsets in $N_n^{[i]}$).

Third, the structure of the formulas (14) and (16), together with the relation

$$(2m - 2)!c_1 = 2^{2m-2}(m - 1)!m!c_m, \quad c_1 = -1/2, \quad (20)$$

imply the following definition of a formal infinite-order differential operator

$$\begin{aligned} \hat{D}(\rho; \gamma) &= \sum_{m \geq 1} \frac{c_m \rho^{2m}}{(2m - 2)!} \left(\frac{\partial}{\partial(\gamma)} \right)^{2m-2} = c_1 \sum_{m=0}^{\infty} \frac{\rho^{2m+2}}{m!(m+1)!} \left(\frac{\partial}{\partial(2\gamma)} \right)^{2m} = \\ &= -\frac{1}{2} \left(\rho^2 + \frac{\rho^4}{8} \frac{\partial^2}{\partial \gamma^2} + \frac{\rho^6}{192} \frac{\partial^4}{\partial \gamma^4} + \dots \right) \end{aligned} \quad (21)$$

branching points), thus signalling about some hidden symmetry of the problem and probable short-cuts, overlooked in the presentation below. The first few coefficients c_m are:

$$c_0 = 1, \quad c_1 = -1/2, \quad c_2 = -1/8, \quad c_3 = -1/16, \quad c_4 = -5/128, \quad c_5 = -7/256, \quad c_6 = -21/1024, \dots$$

(note that $\sum_{m=0}^{\infty} \frac{x^{m+1}}{m!(m+1)!} = J_1(2x)$, a modified Bessel function).

Using this new notation, we can rewrite (14) as

$$\int dS_{DV} = \sum_{\epsilon, \bar{\epsilon}} \left(\prod_{l \in \bar{\epsilon}} \hat{D}(\rho_l; \gamma_{il}) \right) \tilde{I}_{\epsilon, \bar{\epsilon}}^{[i]}(\rho_i, \gamma_{i\cdot}), \quad (22)$$

where the integrals

$$\tilde{I}_{\epsilon, \bar{\epsilon}}^{[i]}(\rho_i, \gamma_{i\cdot}) = \int \frac{\prod_{k \in \epsilon} (x + \gamma_{ik})}{\prod_{l \in \bar{\epsilon}} (x + \gamma_{il})} \sqrt{x - \rho_i^2} \, dx \quad (23)$$

are just another representation of (15) with indices m_j taking only two values: 0 (for all $j \in \epsilon$) and 1 (for all $j \in \bar{\epsilon}$).

While (23) is more convenient for the general formulas like (22) notation (15) can still be of more use in concrete applications (for particular n). For example, in the case of $n = 2$ we have just two terms in (22):

n = 2 :

$$\int_1 dS_{DV} = I_0^{(1)}(\rho_1; \gamma_{12}) + \hat{D}(\rho_2; \gamma_{12}) I_1^{(1)}(\rho_1; \gamma_{12}) \quad (24)$$

with

$$\begin{aligned} I_0^{(1)} &= \tilde{I}_{\{2\}; \emptyset}^{(1)} = \int (x + \gamma_{12}) \sqrt{x^2 - \rho_1^2} dx, \\ I_1^{(1)} &= \tilde{I}_{\emptyset; \{2\}}^{(1)} = \int \frac{\sqrt{x^2 - \rho_1^2}}{x + \gamma_{12}} dx, \end{aligned} \quad (25)$$

and in the case of $n = 3$ there are already four terms:

n = 3 :

$$\begin{aligned} \int_1 dS_{DV} &= I_{00}^{(1)}(\rho_1; \gamma_{12}, \gamma_{13}) + \hat{D}(\rho_2; \gamma_{12}) I_{10}^{(1)}(\rho_1; \gamma_{12}, \gamma_{13}) + \\ &+ \hat{D}(\rho_3; \gamma_{13}) I_{01}^{(1)}(\rho_1; \gamma_{12}, \gamma_{13}) + \hat{D}(\rho_2; \gamma_{12}) \hat{D}(\rho_3; \gamma_{13}) I_{11}^{(1)}(\rho_1; \gamma_{12}, \gamma_{13}) \end{aligned} \quad (26)$$

with

$$\begin{aligned} I_{00}^{(1)} &= \tilde{I}_{\{2,3\}; \emptyset}^{(1)} = \int (x + \gamma_{12})(x + \gamma_{13}) \sqrt{x^2 - \rho_1^2} dx, \\ I_{10}^{(1)} &= \tilde{I}_{\{3\}; \{2\}}^{(1)} = \int \frac{x + \gamma_{13}}{x + \gamma_{12}} \sqrt{x^2 - \rho_1^2} dx, \\ I_{01}^{(1)} &= \tilde{I}_{\{2\}; \{3\}}^{(1)} = \int \frac{x + \gamma_{12}}{x + \gamma_{13}} \sqrt{x^2 - \rho_1^2} dx, \\ I_{11}^{(1)} &= \tilde{I}_{\emptyset; \{2,3\}}^{(1)} = \int \frac{\sqrt{x^2 - \rho_1^2}}{(x + \gamma_{12})(x + \gamma_{13})} dx. \end{aligned} \quad (27)$$

2.2 Handling the integrals

In order to evaluate the integrals $\tilde{I}_{\epsilon; \bar{\epsilon}}^{[i]}$ in (23), it is first necessary to transform the *integrands*.

First, the denominator:

$$\prod_{l \in \bar{\epsilon}} \frac{1}{x + \gamma_{il}} = \sum_{l \in \bar{\epsilon}} \frac{r^{(\bar{\epsilon})}(\gamma_{l\cdot})}{x + \gamma_{il}}, \quad (28)$$

where

$$r^{(\bar{\epsilon})}(\gamma_{l\cdot}) \equiv \prod_{l' \in \bar{\epsilon}_l} \frac{1}{\gamma_{ll'}} \quad (29)$$

(let us remind that $\bar{\epsilon}_l$ is obtained from $\bar{\epsilon}$ by throwing away the element $l \in \bar{\epsilon}$).

Second, for each term in the sum (28) we appropriately transform the numerator. There are two rather different ways to do it, leading to two different representations of the integral.

2.2.1 Representation through symmetric functions

One way is to expand the numerator in (23) for the l -th item of the sum (28) in powers of $(x + \gamma_{il})$:

$$\begin{aligned} \prod_{k \in \epsilon} (x + \gamma_{ik}) &= \prod_{k \in \epsilon} ((x + \gamma_{il}) + \gamma_{lk}) = \\ &= \sum_{m=0}^{d(\epsilon)} (x + \gamma_{il})^{d(\epsilon)-m} e_m^{(\epsilon)}(\gamma_{l\cdot}) = \sum_{m=-1}^{d(\epsilon)-1} (x + \gamma_{il})^{m+1} e_{d(\epsilon)-1-m}^{(\epsilon)}(\gamma_{l\cdot}). \end{aligned} \quad (30)$$

Here

$$e_m^{(\epsilon)}(\gamma_{l\cdot}) = \sum_{\substack{k_1 < \dots < k_m \\ k_1, \dots, k_m \in \epsilon}} \gamma_{lk_1} \dots \gamma_{lk_m} \quad (31)$$

are symmetric polynomials, made from the set of γ_{lk} with given $l \in \bar{\epsilon}$ and all $k \in \epsilon$. Sometime it can be more convenient to use "exclusive" notation instead of "inclusive" one: $e_m^{[\bar{\epsilon}]}(\gamma_{l\cdot}) \equiv e_m^{(\epsilon)}(\gamma_{l\cdot})$. (Note that symmetric functions $r^{(\bar{\epsilon})}(\gamma_{l\cdot})$ and $e_m^{(\epsilon)}(\gamma_{l\cdot})$ do not explicitly depend on i).

With the help of these expressions we obtain for the integral (23),

$$I_{0\dots 0}^{[i]}(\rho_i; \gamma_i) = \sum_{m=0}^{n-1} e_{n-m-1}^{[i]}(\gamma_i) J_m(\rho_i; 0). \quad (32)$$

for $\bar{\epsilon} = \emptyset$ and

$$\tilde{I}_{\epsilon, \bar{\epsilon}}^{[i]}(\rho_i, \gamma_{i\cdot}) = \sum_{l \in \bar{\epsilon}} r^{(\bar{\epsilon})}(\gamma_{l\cdot}) \sum_{p=-1}^{d(\epsilon)-1} e_{d(\epsilon)-1-p}^{(\epsilon)}(\gamma_{l\cdot}) J_p(\rho_i; \gamma_{il}), \quad (33)$$

for $\bar{\epsilon} \neq \emptyset$, where the remaining integrals depend on just two parameters:

$$J_p(\rho; \gamma) \equiv \int (x + \gamma)^p \sqrt{x^2 - \rho^2} dx = \sum_{m=0}^p \frac{p!}{m!(p-m)!} \gamma^{p-m} J_m(\rho; 0), \quad \text{for } p \geq 0. \quad (34)$$

Thus

$$\begin{aligned} \int_i dS_{DV} &= I_{0\dots 0}^{[i]}(\rho_i; \gamma_{i\cdot}) + \\ &+ \sum_{\substack{\epsilon, \bar{\epsilon} \\ \bar{\epsilon} \neq \emptyset}} \left(\prod_{l \in \bar{\epsilon}} \hat{D}(\rho_l, \gamma_{il}) \right) \left(\sum_{l \in \bar{\epsilon}} r^{(\bar{\epsilon})}(\gamma_{l\cdot}) \left[\sum_{p=-1}^{d(\epsilon)-1} e_{d(\epsilon)-p-1}^{(\epsilon)}(\gamma_{l\cdot}) J_p(\rho_i; \gamma_{il}) \right] \right) \end{aligned} \quad (35)$$

The disadvantage of eq.(35) is the need to supplement it by (34), which spoils the relatively nice structure of (35): the sum in the square brackets should be actually transformed to

$$\sum_{p=-1}^{d(\epsilon)-1} e_{d(\epsilon)-p-1}^{(\epsilon)}(\gamma_{l\cdot}) J_p(\rho_i; \gamma_{il}) = e_{d(\epsilon)}^{(\epsilon)}(\gamma_{l\cdot}) J_{-1}(\rho_i; \gamma_{il}) + \sum_{m=0}^{d(\epsilon)-1} \varepsilon_{d(\epsilon)-m-1}^{(\epsilon)}(\gamma_{i\cdot} | \gamma_{l\cdot}) J_m(\rho_i; 0) \quad (36)$$

with

$$\varepsilon_{d(\epsilon)-m-1}^{(\epsilon)}(\gamma_{i\cdot} | \gamma_{l\cdot}) = \sum_{p=m}^{d(\epsilon)-1} \frac{p!}{m!(p-m)!} e_{d(\epsilon)-p-1}^{(\epsilon)}(\gamma_{l\cdot}) \gamma_{il}^{p-m} \quad (37)$$

or

$$\varepsilon_m^{(\epsilon)}(\gamma_{i\cdot} | \gamma_{l\cdot}) = \sum_{p=0}^m \frac{(d(\epsilon) - 1 - p)!}{(d(\epsilon) - 1 - m)!(m - p)!} e_p^{(\epsilon)}(\gamma_{l\cdot}) \gamma_{il}^{m-p} \quad (38)$$

However, once the need to introduce the ε -coefficients is accepted, it is better to do it earlier, handling the numerator in (23) in a different way from the very beginning.

2.2.2 Dividing the polynomials

Another way to deal with the numerator in the l -th item of the sum (28) is just to *divide* it by $(x + \gamma_{il})$, obtaining the ratio polynomial of degree $d(\epsilon) - 1$ and the residue:⁵

$$\prod_{k \in \epsilon} (x + \gamma_{ik}) = \sum_{p=0}^{d(\epsilon)} e_{d(\epsilon)-p}^{(\epsilon)}(\gamma_{i\cdot}) x^p = \varepsilon_{-1}^{(\epsilon)}(\gamma_{i\cdot} | \gamma_{l\cdot}) + (x + \gamma_{il}) \sum_{m=0}^{d(\epsilon)-1} \varepsilon_m^{(\epsilon)}(\gamma_{i\cdot} | \gamma_{l\cdot}) x^m \quad (39)$$

The functions $\varepsilon_m^{(\epsilon)}$ are expressed through $e_p^{(\epsilon)}(\gamma_{i\cdot})$ by recurrent relations,

$$\varepsilon_{m-1}^{(\epsilon)} + \gamma_{il} \varepsilon_m^{(\epsilon)} = e_{d(\epsilon)-m}^{(\epsilon)}(\gamma_{i\cdot}), \quad (40)$$

⁵Note that symmetric functions $e^{(\epsilon)}$ at the l.h.s. of (39) have the arguments γ_{ik} , $k \in \epsilon$, while in the previous subsection 2.2.1 their arguments were rather γ_{lk} with $l \in \bar{\epsilon}$. Accordingly, $\gamma_{i\cdot}$ can seem to be a better choice for the argument of ε -functions, if one looks at (39), and the same can seem about $\gamma_{l\cdot}$ if one looks at (38). In fact, the truth is somewhere in between; ε_m with low m are closer to $e(\gamma_{l\cdot})$, while those with high m are closer to $e(\gamma_{i\cdot})$. We use a somewhat heavy notation, with both types of arguments included. It certainly deserves to be improved.

so that

$$\begin{aligned}
\varepsilon_{d(\epsilon)-1}^{(\epsilon)} &= e_0^{(\epsilon)}(\gamma_{i\cdot}) = 1, \\
\varepsilon_{d(\epsilon)-2}^{(\epsilon)} &= e_1^{(\epsilon)}(\gamma_{i\cdot}) - \gamma_{il} e_0^{(\epsilon)}(\gamma_{i\cdot}) = -\gamma_{il} + \sum_{k \in \epsilon} \gamma_{ik}, \\
&\dots \\
\varepsilon_0^{(\epsilon)} &= \frac{\tilde{\Delta}_i^{(\epsilon)} - \tilde{\Delta}_l^{(\epsilon)}}{\gamma_{il}}, \\
\varepsilon_{-1}^{(\epsilon)} &= e_{d(\epsilon)}^{(\epsilon)}(\gamma_{l\cdot}) \equiv \tilde{\Delta}_l^{(\epsilon)}
\end{aligned} \tag{41}$$

The last lines here are expressed through

$$\tilde{\Delta}_l^{(\epsilon)} \equiv \tilde{\Delta}_l^{[i, \bar{\epsilon}]} \equiv \prod_{k \in \epsilon} \gamma_{lk}. \tag{42}$$

For $\varepsilon = \emptyset$ we skip the superscript:

$$\tilde{\Delta}_i \equiv \tilde{\Delta}_i^{[i]} = \prod_{j \neq i} \gamma_{ij}. \tag{43}$$

Tilde means that the products are made from γ 's, notation without tildes are reserved to denote the same quantities, but made from α 's.

In terms of ε functions we obtain instead of (33)

$$\tilde{I}_{\epsilon, \bar{\epsilon}}^{[i]}(\rho_i, \gamma_{i\cdot}) = \varepsilon_{-1}^{(\epsilon)}(\gamma_{i\cdot} | \gamma_{l\cdot}) J_{-1}(\rho_i; \gamma_{il}) + \sum_{m=0}^{d(\epsilon)-1} \varepsilon_{d(\epsilon)-m-1}^{(\epsilon)}(\gamma_{i\cdot} | \gamma_{l\cdot}) J_m(\rho_i; 0) \tag{44}$$

for $\bar{\epsilon} \neq \emptyset$, and, finally

$$\begin{aligned}
\int_i dS_{DV} &= I_{0\dots 0}^{[i]}(\rho_i; \gamma_{i\cdot}) + \\
&+ \sum_{\substack{\epsilon, \bar{\epsilon} \\ \bar{\epsilon} \neq \emptyset}} \left(\prod_{l \in \bar{\epsilon}} \hat{D}(\rho_l, \gamma_{il}) \right) \left(\sum_{l \in \bar{\epsilon}} r^{(\bar{\epsilon})}(\gamma_{l\cdot}) \left[\varepsilon_{-1}^{(\epsilon)}(\gamma_{i\cdot} | \gamma_{l\cdot}) J_{-1}(\rho_i; \gamma_{il}) + \sum_{m=0}^{d(\epsilon)-1} \varepsilon_{d(\epsilon)-m-1}^{(\epsilon)}(\gamma_{i\cdot} | \gamma_{l\cdot}) J_m(\rho_i; 0) \right] \right)
\end{aligned} \tag{45}$$

The price for obtaining a representation through the simple integrals $J_m(\rho_i; 0)$ with vanishing second argument is the appearance of somewhat sophisticated coefficients $\varepsilon(\gamma)$. The γ -dependence can *not* be eliminated from the integral $J_{-1}(\rho; \gamma)$, but of course it can be easily evaluated, see the next subsection 2.3.

2.2.3 Examples of $n = 2$ and $n = 3$

The simple example integrals in (25) and (27) are in fact not sensitive to all these complications and are easily expressed through J_m -integrals.

n = 2 :

$$\begin{aligned}
I_0^{(1)}(\rho_1; \gamma_{12}) &= J_1(\rho_1; 0) + \gamma_{12} J_0(\rho_1; 0), \\
I_1^{(1)}(\rho_1; \gamma_{12}) &= J_{-1}(\rho_1; \gamma_{12}),
\end{aligned} \tag{46}$$

and
n = 3 :

$$\begin{aligned}
I_{00}^{(1)}(\rho_1; \gamma_{12}, \gamma_{13}) &= J_2(\rho_1; 0) + (\gamma_{12} + \gamma_{13}) J_1(\rho_1; 0) + \gamma_{12} \gamma_{13} J_0(\rho_1; 0), \\
I_{10}^{(1)}(\rho_1; \gamma_{12}, \gamma_{13}) &= J_0(\rho_1; 0) + \gamma_{23} J_{-1}(\rho_1; \gamma_{12}), \\
I_{01}^{(1)}(\rho_1; \gamma_{12}, \gamma_{13}) &= J_0(\rho_1; 0) - \gamma_{23} J_{-1}(\rho_1; \gamma_{13}), \\
I_{11}^{(1)}(\rho_1; \gamma_{12}, \gamma_{13}) &= \frac{1}{\gamma_{23}} [J_{-1}(\rho_1; \gamma_{12}) - J_{-1}(\rho_1; \gamma_{13})]
\end{aligned} \tag{47}$$

2.3 J_m -integrals and their periods

These elementary integrals are given by somewhat heavy-looking formulas:⁶

$$\begin{aligned}
J_{2m}(\rho; 0) &= c_{m+1} \rho^{2m+2} t + \rho^{2m+2} \sum_{j=1}^{m+1} c_{m+1}^{(j)} \sinh 2jt, \quad m \geq 0, \\
J_{2m-1}(\rho; 0) &= \rho^{2m+1} \sum_{j=0}^m c_{m+1/2}^{(j+1/2)} \sinh(2j+1)t, \quad m > 0, \\
J_{-1}(\rho; \gamma) &= \sqrt{x^2 - \rho^2} - \gamma t + \sqrt{\gamma^2 - \rho^2} \log \frac{\rho z + \gamma - \sqrt{\gamma^2 - \rho^2}}{\rho z + \gamma + \sqrt{\gamma^2 - \rho^2}},
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
c_k^{(l)} &\equiv c_k^{[k-l]} \equiv 2^{1-2k} \frac{2l^2 - k}{l} \cdot \frac{(2k-2)!}{(k-l)!(k+l)!}, \\
c_k^{(k)} &= c_k^{[0]} = \frac{1}{2^{2k} k}, c_k^{(k-1)} = c_k^{[1]} = \frac{2(k-2)}{2^{2k}(k-1)}, c_k^{(k-2)} = c_k^{[2]} = \frac{2k^2 - 9k + 8}{2^{2k}(k-2)}, \dots, \\
&\text{as } l \rightarrow 0, \quad c_k^{(l)} \sim \frac{c_k}{l},
\end{aligned} \tag{49}$$

and

$$x = \rho \cosh t, \quad z = e^t = \frac{x + \sqrt{x^2 - \rho^2}}{\rho}. \tag{50}$$

⁶For example,

$$\begin{aligned}
J_0 &= \rho^2 \int \sinh^2 t dt = \frac{\rho^2}{4} (\sinh 2t - 2t), \\
J_1 &= \rho^3 \int \cosh t \sinh^2 t dt = \frac{\rho^3}{12} (\sinh 3t - 3 \sinh t), \\
J_2 &= \rho^4 \int \cosh^2 t \sinh^2 t dt = \frac{\rho^4}{32} (\sinh 4t - 4t), \\
&\dots
\end{aligned}$$

Actually, we are interested in two types of integration paths:

A_i : x from ρ_i to $-\rho_i$, i.e. t from 0 to $i\pi$ and z from 1 to -1 , and

B_i : x from ρ to $\Lambda_i \equiv \Lambda_{(\gamma_i)} \equiv \Lambda - \gamma_i$, i.e. z from 1 to Z_i/ρ_i , where

$$Z_{(\gamma)}(\rho) = \Lambda_{(\gamma)} + \sqrt{\Lambda_{(\gamma)}^2 - \rho^2} = 2\Lambda_{(\gamma)} + \sum_{m=1}^{\infty} c_m \frac{\rho^{2m}}{\Lambda_{(\gamma)}^{2m-1}}, \quad (51)$$

and t from 0 to $\log Z_i/\rho_i$.

We denote the integrals J_m along these paths by J_m^A and J_m^B . The A_i and B_i periods of dS_{DV} , i.e. $-S_i/g_{n+1}$ and Π_i/g_{n+1} are then obtained by substitution of the corresponding integrals, into (14) and multiplying by an extra factor of 2. Note, that all the operators $\hat{D}(\gamma_l; \gamma_{il})$ in (45) commute with Λ_i and Z_i : derivatives act on γ_l , *not* on γ_i ,

For the integrals along the A contour we obtain simple expressions:

$$\begin{aligned} J_{2m}^A(\rho; 0) &= i\pi c_{m+1} \rho^{2m+2}, \quad m \geq 0, \\ J_{2m-1}^A(\rho; 0) &= 0, \quad m > 0, \\ J_{-1}^A(\rho, \gamma) &= -i\pi(\gamma - \sqrt{\gamma^2 - \rho^2}) = i\pi \sum_{k=1}^{\infty} c_k \frac{\rho^{2k}}{\gamma^{2k-1}} = \\ &= -i\pi \left(\frac{\rho^2}{2\gamma} + \frac{\rho^4}{8\gamma^3} + \frac{\rho^6}{16\gamma^5} + \dots \right) \end{aligned} \quad (52)$$

but those for the B contour are more sophisticated (see (49) for the definition of $c_k^{[l]}$):

$$\begin{aligned} J_{2m}^B(\rho; Z) &= \frac{i}{\pi} J_{2m}^A(\rho; 0) \log \frac{\rho}{2\Lambda} + \frac{1}{2} \sum_{j=0}^m c_{m+1}^{[j]} \rho^{2j} Z^{2m+2-2j} + O(\Lambda^{-1}), \\ J_{2m-1}^B(\rho; Z) &= \frac{1}{2} \sum_{j=0}^m c_{m+1/2}^{[j]} \rho^{2j} Z^{2m+1-2j} + O(\Lambda^{-1}), \\ J_{-1}^B(\rho, \gamma | \Lambda_{(\gamma')}) &= \Lambda_{(\gamma')} - \gamma \log \frac{2\Lambda}{\rho} + \sqrt{\gamma^2 - \rho^2} L(\rho; \gamma) = \\ &= \Lambda_{(\gamma')} + \gamma \log \frac{\gamma}{\Lambda} + \frac{i}{\pi} J_{-1}^A(\rho, \gamma) \log \frac{\rho}{2\gamma} + \sum_{k,l=0}^{\infty} \frac{k+2}{k+1} c_l c_{k+2} \frac{\rho^{2k+2l+2}}{\gamma^{2k+2l+1}} = \\ &= \Lambda_{(\gamma')} + \gamma \log \frac{\gamma}{\Lambda} + \frac{i}{\pi} J_{-1}^A \log \frac{\rho}{2\gamma} - \frac{\rho^2}{4\gamma} + \frac{\rho^4}{32\gamma^3} + \frac{5\rho^6}{192\gamma^5} + \frac{59\rho^8}{3 \cdot 1024\gamma^7} + \dots \end{aligned} \quad (53)$$

Here

$$\begin{aligned} \frac{\partial L(\rho, \gamma)}{\partial \gamma} &= \frac{1}{\sqrt{\gamma^2 - \rho^2}} = \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{k! \Gamma(1/2)} \frac{\rho^{2k}}{\gamma^{2k-1}} = -2 \sum_{k=1}^{\infty} (k+1) c_{k+1} \frac{\rho^{2k}}{\gamma^{2k-1}}, \\ L(\rho; \gamma) &= \log \frac{\rho + \gamma + \sqrt{\gamma^2 - \rho^2}}{\rho + \gamma - \sqrt{\gamma^2 - \rho^2}} = \log \frac{2\gamma}{\rho} + \sum_{k=1}^{\infty} \frac{(k+1) c_{k+1}}{k} \frac{\rho^{2k}}{\gamma^{2k}}, \\ \sqrt{\gamma^2 - \rho^2} L(\rho; \gamma) &= \sqrt{\gamma^2 - \rho^2} \log \frac{2\gamma}{\rho} + \sum_{k,l=0}^{\infty} \frac{(k+2) c_{k+2} c_l}{k} \frac{\rho^{2k+2l+2}}{\gamma^{2k+2l+1}} \end{aligned} \quad (54)$$

3 The first three terms of the ρ^2 -expansion

3.1 Truncations and S -integrals

Let us now pick up the first terms of expansion in powers of ρ^2 in (45). Since $S = O(\rho^2)$, for the purpose of calculating $\mathcal{F}_3(S|\alpha)$ it is enough to keep the first three terms, of orders ρ^0 , ρ^2 and ρ^4 . This implies that $d(\bar{\epsilon}) \leq 2$ and leaves only three options for $\bar{\epsilon}$: $\bar{\epsilon} = \emptyset$, $\bar{\epsilon} = \{l\}$ and $\bar{\epsilon} = \{l_1, l_2\}$, so that (45) is truncated to

$$\begin{aligned} \int_i dS_{DV} = & \sum_{m=0}^{n-1} e_{n-m-1}^{[i]}(\gamma_{i\cdot}) J_m(\rho_i; 0) - \\ & - \frac{1}{2} \sum_{l \neq i} \left(\rho_l^2 + \frac{\rho_l^4}{8} \frac{\partial^2}{\partial \gamma_l^2} \right) \left[\varepsilon_{-1}^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot}) J_{-1}(\rho_i; \gamma_{il}) + \sum_{m=0}^{n-3} \varepsilon_m^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot}) J_m(\rho_i; 0) \right] + \\ & + \frac{1}{4} \sum_{\substack{l_1 < l_2 \\ l_1, l_2 \neq i}} \frac{\rho_{l_1}^2 \rho_{l_2}^2}{\gamma_{l_1 l_2}} \left[\varepsilon_{-1}^{[il_1 l_2]}(\gamma_{i\cdot} | \gamma_{l_1 \cdot}) J_{-1}(\rho_i; \gamma_{il_1}) - \varepsilon_{-1}^{[il_1 l_2]}(\gamma_{i\cdot} | \gamma_{l_2 \cdot}) J_{-1}(\rho_i; \gamma_{il_2}) + \right. \\ & \left. + \sum_{m=0}^{n-4} \left(\varepsilon_m^{[il_1 l_2]}(\gamma_{i\cdot} | \gamma_{l_1 \cdot}) J_m(\rho_i; 0) - \varepsilon_m^{[il_1 l_2]}(\gamma_{i\cdot} | \gamma_{l_2 \cdot}) J_m(\rho_i; 0) \right) \right] \end{aligned} \quad (55)$$

The sums over m could also be truncated, if there were no Λ_i -dependent terms in J_m^B . However, for A -integrals there is no such problem, and only terms with $m \leq 2$ in the first line and with $m = 0$ in the second one can contribute, because $J_m^A = O(\rho^{m+2})$. Moreover, since also $J_{-1}^A = O(\rho^2)$ (note, that this is *not* true for J_{-1}^B , even if the Λ_i -dependent terms were excluded), the last term in (55) is also irrelevant, and we get for the A_i -period S_i of dS_{DV} :

$$\begin{aligned} -\frac{1}{2\pi i g_{n+1}} S_i = & -\frac{1}{2} e_{n-1}^{[i]}(\gamma_{i\cdot}) \rho_i^2 - \frac{1}{8} e_{n-3}^{[i]}(\gamma_{i\cdot}) \rho_i^4 + \\ & + \frac{1}{4} \sum_{l \neq i} \rho_i^2 \rho_l^2 \left(\frac{\varepsilon_{-1}^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot})}{\gamma_{li}} + \varepsilon_0^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot}) \right) + O(\rho^6), \end{aligned} \quad (56)$$

so that for $\hat{S}_i \equiv \frac{S_i}{i\pi g_{n+1}}$ and

$$\begin{aligned} e_{n-1}^{[i]}(\gamma_{i\cdot}) &= \prod_{j \neq i} \gamma_{ij} \equiv \tilde{\Delta}_i, \\ e_{n-3}^{[i]}(\gamma_{i\cdot}) &= \tilde{\Delta}_i \sum_{\substack{j < k \\ j, k \neq i}} \frac{1}{\gamma_{ij} \gamma_{ik}}, \\ \varepsilon_{-1}^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot}) &= \prod_{k \neq i, l} \gamma_{ki} \equiv \tilde{\Delta}_l^{[il]} = \frac{\tilde{\Delta}_l}{\gamma_{li}}, \\ \varepsilon_0^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot}) &= \frac{\tilde{\Delta}_i^{[il]} - \tilde{\Delta}_l^{[il]}}{\gamma_{il}} = \frac{\tilde{\Delta}_i + \tilde{\Delta}_l}{\gamma_{il}^2} \end{aligned} \quad (57)$$

we get:

$$\begin{aligned}
\hat{S}_i &= \rho_i^2 \tilde{\Delta}_i + \frac{1}{4} \rho_i^4 \tilde{\Delta}_i \sum_{\substack{j < k \\ j, k \neq i}} \frac{1}{\gamma_{ij} \gamma_{ik}} - \frac{1}{2} \rho_i^2 \sum_{l \neq i} \left(-\frac{\tilde{\Delta}_l}{\gamma_{li}^2} + \frac{\tilde{\Delta}_i + \tilde{\Delta}_l}{\gamma_{il}^2} \right) = \\
&= \rho_i^2 \tilde{\Delta}_i \left(1 + \frac{1}{4} \rho_i^2 \sum_{\substack{j < k \\ j, k \neq i}} \frac{1}{\gamma_{ij} \gamma_{ik}} - \frac{1}{2} \sum_{j \neq i} \frac{\rho_j^2}{\gamma_{ij}^2} + O(\rho^4) \right)
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
\rho_i^2 &= \frac{\hat{S}_i}{\tilde{\Delta}_i} \left(1 + \sum_{j=1}^n \zeta_{ij}(\gamma) \hat{S}_j + O(S^2) \right), \\
\zeta_{ii} &= -\frac{1}{4\tilde{\Delta}_i} \sum_{\substack{j < k \\ j, k \neq i}} \frac{1}{\gamma_{ij} \gamma_{ik}}, \\
\zeta_{ij} &= -\frac{1}{\gamma_{ij}^2 \tilde{\Delta}_j}, \quad j \neq i.
\end{aligned} \tag{59}$$

Remarkably, due to peculiar relation⁷

$$\hat{S}_i \log(\Delta^{[i]} \rho_i^2) - (\Delta^{[i]} \rho_i^2) = \hat{S}_i \log \hat{S}_i - \hat{S}_i + O(S^3), \tag{60}$$

the coefficients ζ from (59) do not show up in explicit expression for $\mathcal{F}_3(S|\alpha)$, though the entire eq.(56) is needed to handle the logarithmic contributions to the prepotential.

3.2 Π -integral

Let us now proceed to evaluation of the Π_i integral:

$$\begin{aligned}
\frac{\Pi_i}{2g_{n+1}} &= \sum_{m \geq 0}^{n-1} e_{n-m-1}^{[i]}(\gamma_{i\cdot}) J_m^B(\rho_i; 0) - \\
&- \frac{1}{2} \sum_{l \neq i} \left(\rho_l^2 + \frac{\rho_l^4}{8} \frac{\partial^2}{\partial \gamma_l^2} \right) \left[\varepsilon_{-1}^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot}) J_{-1}^B(\rho_i; \gamma_{il}) + \sum_{m=0}^{n-3} \varepsilon_m^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot}) J_m^B(\rho_i; 0) \right] + \\
&+ \frac{1}{4} \sum_{\substack{l_1 < l_2 \\ l_1, l_2 \neq i}} \frac{\rho_{l_1}^2 \rho_{l_2}^2}{\gamma_{l_1 l_2}} \left[\varepsilon_{-1}^{[il_1 l_2]}(\gamma_{i\cdot} | \gamma_{l_1 \cdot}) J_{-1}^B(\rho_i; \gamma_{il_1}) - \varepsilon_{-1}^{[il_1 l_2]}(\gamma_{i\cdot} | \gamma_{l_2 \cdot}) J_{-1}^B(\rho_i; \gamma_{il_2}) \right. \\
&\quad \left. + \sum_{m=0}^{n-4} \left(\varepsilon_m^{[il_1 l_2]}(\gamma_{i\cdot} | \gamma_{l_1 \cdot}) J_m^B(\rho_i; 0) - \varepsilon_m^{[il_1 l_2]}(\gamma_{i\cdot} | \gamma_{l_2 \cdot}) J_m^B(\rho_i; 0) \right) \right] + \\
&\quad + O(\rho^6)
\end{aligned} \tag{61}$$

where expressions for J_m^B should be substituted from (53). After that the r.h.s. consists of the contributions of three essentially different types: containing Λ_i , containing logarithms (of any kind of arguments, Λ , ρ and γ), and power-like terms, proportional to ρ^2 and ρ^4 with γ -dependent coefficients.

⁷The combination $S \log S - S = \frac{1}{2} \partial_S [S^2 (\log S - \frac{3}{2})]$ is characterized by the fact that its S -derivative is exactly $\log S$.

In the rest of this *subsection* we analyze these three types of terms separately, giving some details (one can easily understand how things work, looking through explicit examples in Appendix B).

3.2.1 Λ -dependent terms

The Λ_i -dependent terms are:

$$\begin{aligned}
& \frac{1}{2} \sum_{m=0}^{n-1} e_{n-m-1}^{[i]}(\gamma_{i\cdot}) \left[c_{m/2+1}^{[0]}(2\Lambda_i)^{m+2} \left(1 - \frac{(m+2)\rho_i^2}{4\Lambda_i^2} + \frac{(m-1)(m+2)\rho_i^4}{32\Lambda_i^4} \right) + \right. \\
& \quad \left. + \theta(m \geq 1) c_{m/2+1}^{[1]} \rho_i^2 (2\Lambda_i)^m \left(1 - \frac{m\rho_i^2}{4\Lambda_i^2} \right) + \theta(m \geq 3) c_{m/2+1}^{[2]} \rho_i^4 (2\Lambda_i)^{m-2} \right] - \\
& - \frac{1}{4} \sum_{l \neq i} \rho_l^2 \sum_{m=-1}^{n-3} \varepsilon_m^{[il]}(\gamma_{i\cdot}|\gamma_{l\cdot}) \left[c_{m/2+1}^{[0]}(2\Lambda_i)^{m+2} \left(1 - \frac{(m+2)\rho_i^2}{4\Lambda_i^2} \right) + \theta(m \geq 1) c_{m/2+1}^{[1]} \rho_i^2 (2\Lambda_i)^m \right] \\
& + \frac{1}{8} \sum_{\substack{l_1 < l_2 \\ l_1, l_2 \neq i}} \frac{\rho_{l_1}^2 \rho_{l_2}^2}{\gamma_{l_1 l_2}} \frac{1}{2} \sum_{m=0}^{n-4} \left[\varepsilon_m^{[il_1 l_2]}(\gamma_{i\cdot}|\gamma_{l\cdot}) - \varepsilon_m^{[il_1 l_2]}(\gamma_{i\cdot}|\gamma_{l\cdot}) \right] c_{m/2+1}^{[0]}(2\Lambda_i)^{m+2} + \\
& \quad + O(\rho^6)
\end{aligned} \tag{62}$$

The relevant coefficients $c_k^{[l]}$ are listed in (49),

$$\frac{1}{2} c_{m/2+1}^{[0]} = \frac{1}{2^{m+2}(m+2)}, \quad \frac{1}{2} c_{m/2+1}^{[1]} = \frac{1}{2^{m+2}} \frac{m-2}{m}, \quad \frac{1}{2} c_{m/2+1}^{[2]} = \frac{1}{2^{m+2}} \frac{m^2 - 5m + 2}{2(m-2)}. \tag{63}$$

An extra coefficient $1/2$ as compared to (61) appears in these formulas because $J_m^B \sim \frac{1}{2} Z_i^{m+2}$ for large Z_i . This is not true for J_{-1}^B , but instead

$$\frac{1}{2} c_{m/2+1}^{[0]}(2\Lambda_i)^{m+2} = \frac{1}{m+2} \Lambda_i^{m+2} \tag{64}$$

is exactly Λ_i for $m = -1$, so the term with $m = -1$ is naturally included into the sum at the r.h.s. of (62). We have introduced the Heaviside step function: $\theta(m \geq k) = 0$ for $m < k$ and $\theta(m \geq k) = 1$ for $m \geq k$.

It is especially easy to see what happens to the ρ -independent terms. By definition of symmetric polynomial $e_m(\gamma_{i\cdot})$,

$$\sum_{m=0}^{n-1} e_{n-m-1}^{[i]}(\gamma_{i\cdot}) \Lambda_i^m = \prod_{j \neq i}^n (\Lambda_i + \gamma_{ij}) = \prod_{j \neq i}^n (\Lambda - \gamma_j) = \frac{1}{\Lambda_i} \tilde{P}_n(\Lambda) \tag{65}$$

with

$$\tilde{P}_n(x) = \prod_{j=1}^n (x - \gamma_j). \tag{66}$$

It remains to multiply both sides of (65) by an extra $\Lambda_i = \Lambda - \gamma_i$ and integrate over Λ_i to get for the leading term in (62):

$$\sum_{m=0}^{n-1} e_{n-m-1}^{[i]}(\gamma_{i\cdot}) \frac{(2\Lambda_i)^{m+2}}{2^{m+2}(m+2)} = \tilde{W}_{n+1}(\Lambda) - \tilde{W}_{n+1}(\gamma_i) \quad (67)$$

with $\tilde{W}'_{n+1}(x) = \tilde{P}_n(x)$. To check this reasoning it is enough to take the Λ -derivative of eq.(67): it obviously gives eq.(65) times Λ_i . If one proceeds in this way, the integration constant at the r.h.s. of (67) is equal to $-\tilde{W}_{n+1}(\gamma_i)$ in order to make the r.h.s. vanishing at $\Lambda = \gamma_i$.

The role of the terms with both Λ_i and ρ^2 -dependence is to change $\tilde{W}_{n+1}(x)$ with γ -dependent coefficients for $W_{n+1}(x)$ with those, made from α 's. This is a small correction, since $\sigma_i = \gamma_i - \alpha_i = O(\rho^2)$ and thus $\tilde{P}_n(x) - P_n(x) = O(\rho^2)$ and $\tilde{W}_{n+1}(x) - W_{n+1}(x) = O(\rho^2)$. Note, that for a given power of Λ_i there is only a single power of ρ^2 that contributes: no infinite series in powers of ρ^2 are involved into the relation between $\tilde{W}_{n+1}(x)$ and $W_{n+1}(x)$. It happens so because this relation actually contains ρ only in the form of the sum rules (100). Just for the same reason one does not need to know higher corrections to (110), though at the first glance they could seem to be important: the sum rules (100) can be applied directly. The simplest way to check all these statements is again to compare the Λ -derivative of (62) with that of $W_{n+1}(\Lambda)$, which is equal to

$$\begin{aligned} W'_{n+1}(\Lambda) &= \prod_{j=1}^n (\Lambda - \alpha_j) = \prod_{j=1}^n (\Lambda - \alpha_i + \alpha_{ij}) = (\Lambda_i + \sigma_i) \prod_{j \neq i} [(\Lambda_i + \sigma_i) + \alpha_{ij}] = \\ &= \sum_{m=0}^{n-1} e_{n-m-1}^{[i]}(\alpha_{i\cdot}) (\Lambda_i + \sigma_i)^{m+1} = \\ &= \sum_{m=0}^{n-1} e_{n-m-1}^{[i]}(\alpha_{i\cdot}) \Lambda_i^{m+1} + \sigma_i \sum_{m=0}^{n-1} (m+1) e_{n-m-1}^{[i]}(\alpha_{i\cdot}) \Lambda_i^m + \\ &\quad + \frac{\sigma_i^2}{2} \sum_{m=0}^{n-1} m(m+1) e_{n-m-1}^{[i]}(\alpha_{i\cdot}) \Lambda_i^{m-1} + O(\rho^6) \end{aligned} \quad (68)$$

For example, in the order $O(\rho^4)$ we have:

$$\begin{aligned} \sum_{m=0}^{n-1} e_{n-m-1}^{[i]}(\gamma_{i\cdot}) \left(\Lambda_i^{m+1} - \frac{\rho_i^2 \Lambda_i^{m-1}}{2} \theta(m \geq 1) \right) - \frac{1}{2} \sum_{l \neq i} \rho_l^2 \sum_{m=0}^{n-2} \varepsilon_{m-1}^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot}) \Lambda_i^m = \\ = \sum_{m=0}^{n-1} e_{n-m-1}^{[i]}(\alpha_{i\cdot}) \Lambda_i^{m+1} + \sigma_i \sum_{m=0}^{n-1} (m+1) e_{n-m-1}^{[i]}(\alpha_{i\cdot}) \Lambda_i^m + O(\rho^4) \end{aligned} \quad (69)$$

Collecting contributions with the same powers of Λ_i , we get:

$$e_0^{[i]}(\gamma_{i\cdot}) = e_0^{[i]}(\alpha_{i\cdot}), \quad i.e. \quad 1 = 1;$$

$$e_1^{[i]}(\gamma_{i\cdot}) = e_1^{[i]}(\alpha_{i\cdot}) + n\sigma_i e_0^{[i]}(\alpha_{i\cdot}), \quad i.e. \quad \sum_{j \neq i} \gamma_{ij} = \sum_{j \neq i} \alpha_{ij} + n\sigma_i;$$

$$\begin{aligned} e_2^{[i]}(\gamma_{i\cdot}) - \frac{\rho_i^2 e_0^{[i]}(\gamma_{i\cdot})}{2} - \frac{1}{2} \sum_{l \neq i} \rho_l^2 \varepsilon_{n-3}^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot}) &= e_2^{[i]}(\alpha_{i\cdot}) + (n-1) e_1^{[i]}(\alpha_{i\cdot}) \sigma_i, \\ i.e. \quad \sum_{\substack{j < k \\ j, k \neq i}} \gamma_{ij} \gamma_{ik} - \frac{1}{2} \sum_{j=1}^n \rho_j^2 &= \sum_{\substack{j < k \\ j, k \neq i}} \alpha_{ij} \alpha_{ik} + (n-1) \sigma_i \sum_{j \neq i} \alpha_{ij}; \\ &\dots \end{aligned} \tag{70}$$

The next step is to express $e_m^{[i]}(\gamma_{i\cdot})$ through γ_i and $e_k(\gamma)$ (or $p_k(\gamma)$) and $e_m^{[i]}(\alpha_{i\cdot})$ through α_i and $e_k(\alpha)$ (or $p_k(\alpha)$). For example,

$$\begin{aligned} e_2^{[i]}(\alpha_{i\cdot}) &= \sum_{\substack{j < k \\ j, k \neq i}} \alpha_{ij} \alpha_{ik} = \\ &= \frac{1}{2} \left(\sum_{j=1}^n \alpha_j \right)^2 - \frac{1}{2} \sum_{j=1}^n \alpha_j^2 - (n-1) \alpha_i \sum_{n=1}^j \alpha_j + \frac{n(n-1)}{2} \alpha_i^2 = \\ &= \frac{p_1^2(\alpha)}{2} - \frac{p_2(\alpha)}{2} - (n-1) \alpha_i p_1(\alpha) + \frac{n(n-1)}{2} \alpha_i^2 \end{aligned} \tag{71}$$

After that eqs.(70) are easily reduced to the sum rules (99), relating $e_m(\alpha)$ and $e_m(\gamma)$. We do not go in further details of this calculation. A more concise way to make and present it in full generality remains to be found.

Of more interest for us below are a few terms with $-1 \leq m \leq 2$, which do not depend on Λ_i at all. They drop out after when the Λ -derivative is taken and do not contribute to above calculation. Instead they will contribute to \mathcal{F}_3 . There are exactly three such terms, coming from integrals J_2^B and J_0^B , the relevant coefficients are $c_2^{[0]} = c_2^{(2)} = 1/32$ and $c_1^{[0]} = c_1^{(1)} = 1/4$:

$$-\frac{1}{4} e_{n-1}^{[i]}(\gamma_{i\cdot}) \rho_i^2 + \frac{1}{32} e_{n-3}^{[i]}(\gamma_{i\cdot}) \rho_i^4 + \frac{1}{8} \rho_i^2 \sum_{l \neq i} \varepsilon_0^{[il]}(\gamma_{i\cdot} | \gamma_{l\cdot}) \rho_l^2 \tag{72}$$

3.2.2 Logarithmic terms

Logarithmic terms are easy to collect. All such terms, coming from J_{2m}^B with $m \geq 0$ are equal to the corresponding J_{2m}^A , multiplied by $\log \frac{\rho_i}{2\Lambda}$, which commutes with $\hat{D}(\rho_i; \gamma_{il})$. Thus they combine altogether to provide $\frac{i}{\pi} (S_i/2g_{n+1}) = \frac{1}{2} \hat{S}_i$, multiplied by the same logarithm. One should take care only of the contributions from

$$\begin{aligned}
J_{-1}^B &= \frac{i}{\pi} J_{-1}^A \left(\log \frac{\rho}{2\Lambda} - \log \frac{\gamma}{\Lambda} \right) + \gamma \log \frac{\gamma}{\Lambda} + \text{non-logarithmic terms} = \\
&= \frac{i}{\pi} J_{-1}^A \log \frac{\rho}{2\Lambda} + \left(\gamma - \frac{\rho^2}{2\gamma} \right) \log \frac{\gamma}{\Lambda} + O(\rho^4) + \text{non-log terms},
\end{aligned} \tag{73}$$

where the first term at the r.h.s. is combined with the other $J_{2m}^A \log \frac{\rho}{2\Lambda}$, and only the second one requires further consideration. Keeping all this in mind, we get for the

$$\begin{aligned}
&\log - \text{terms in } \frac{\Pi_i}{2g_{n+1}} = \frac{\hat{S}_i}{2} \log \frac{\rho_i}{2\Lambda} - \\
&-\frac{1}{2} \sum_{l \neq i} \left[\left(\rho_l^2 + \frac{\rho_l^4}{8} \frac{\partial^2}{\partial \gamma_l^2} \right) \varepsilon_{-1}^{[il]}(\gamma_i | \gamma_l) \left(\gamma_{il} - \frac{\rho_i^2}{2\gamma_{il}} \right) \right] \log \frac{\gamma_{il}}{\Lambda} + \\
&+ \frac{1}{4} \sum_{\substack{l_1 < l_2 \\ l_1, l_2 \neq i}} \frac{\rho_{l_1}^2 \rho_{l_2}^2}{\gamma_{l_1 l_2}} \left(\varepsilon_{-1}^{[il_1 l_2]}(\gamma_i | \gamma_{l_1}) \gamma_{il_1} \log \frac{\gamma_{il_1}}{\Lambda} - \varepsilon_{-1}^{[il_1 l_2]}(\gamma_i | \gamma_{l_2}) \gamma_{il_2} \log \frac{\gamma_{il_2}}{\Lambda} \right) + O(\rho^6) = \\
&= \frac{\hat{S}_i}{2} \log \frac{\rho_i}{2\Lambda} + \frac{1}{2} \sum_{l \neq i} \left[\left(\rho_l^2 + \frac{\rho_l^4}{8} \frac{\partial^2}{\partial \gamma_l^2} \right) \tilde{\Delta}_l \left(1 - \frac{\rho_i^2}{2\gamma_{il}^2} \right) \right] \log \frac{\gamma_{il}}{\Lambda} - \\
&- \frac{1}{4} \sum_{l \neq i} \rho_l^2 \tilde{\Delta}_l \log \frac{\gamma_{il}}{\Lambda} \sum_{k \neq i, l} \frac{\rho_k^2}{\gamma_{lk}^2} + O(\rho^6) =
\end{aligned} \tag{74}$$

$$\begin{aligned}
&= \frac{\hat{S}_i}{2} \log \frac{\rho_i}{2\Lambda} + \\
&+ \frac{1}{2} \sum_{l \neq i} \rho_l^2 \tilde{\Delta}_l \left(1 + \frac{\rho_l^2}{4} \sum_{\substack{k_1 < k_2 \\ k_1, k_2 \neq l}} \frac{1}{\gamma_{lk_1} \gamma_{lk_2}} - \frac{1}{2} \frac{\rho_i^2}{\gamma_{li}^2} - \frac{1}{2} \sum_{k \neq i, l} \frac{\rho_k^2}{\gamma_{lk}^2} + O(\rho^6) \right) \log \frac{\gamma_{il}}{\Lambda} =
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(58)}{=} \frac{\hat{S}_i}{4} \log \frac{\rho_i^2}{4} - \frac{\hat{S}_i}{2} \log \Lambda + \frac{1}{2} \sum_{j \neq i} \hat{S}_j \log \frac{\gamma_{ij}}{\Lambda} + O(S^3) = \\
&= -\frac{1}{2} \left(\sum_{j=1}^n \hat{S}_j \right) \log \Lambda + \frac{\hat{S}_i}{4} \log \frac{\rho_i^2 \Delta_i}{4} - \frac{1}{4} \sum_{j \neq i} (\hat{S}_i - 2\hat{S}_j) \log \gamma_{ij}
\end{aligned}$$

We see that the terms of order ρ^4 are nicely combined to correct $\rho_j^2 \tilde{\Delta}_j$ into \hat{S}_j , according to (58). The same will, of course, happen in all higher orders of expansion in powers of ρ^2 , so that the answer at the r.h.s. of (74) is in fact exact (but our calculation is true only up to the $O(S^3)$ terms). The first term at the r.h.s. of (74) with $\log \rho_i^2$ is combined with (75) from the next subsection to give an expression entirely in terms of S , according to eq.(60).

3.2.3 Polynomial terms

The sources of the relevant (for calculation of the cubic piece in the prepotential) terms in $\Pi_i/2g_{n+1}$ are:

– the Λ_i -independent contributions (72) from terms with Z_i (such contributions exist, because $Z_i \neq \Lambda_i$) – these give

$$-\frac{1}{4}e_{n-1}^{[i]}(\gamma_{i\cdot})\rho_i^2 = -\frac{\rho_i^2\tilde{\Delta}_i}{4} \quad (75)$$

in the order $O(\rho^2)$ and

$$\begin{aligned} & \frac{1}{32}e_{n-3}^{[i]}(\gamma_{i\cdot})\rho_i^4 + \frac{1}{8}\rho_i^2 \sum_{l \neq i} \varepsilon_0^{[il]}(\gamma_{i\cdot}|\gamma_{l\cdot})\rho_l^2 = \\ & = \frac{\rho_i^4\tilde{\Delta}_i}{32} \sum_{\substack{j < k \\ j, k \neq i}} \frac{1}{\gamma_{ij}\gamma_{ik}} + \frac{1}{8} \sum_{l \neq i} \frac{\rho_i^2\rho_l^2(\tilde{\Delta}_i + \tilde{\Delta}_l)}{\gamma_{il}^2} \end{aligned} \quad (76)$$

in the order $O(\rho^4)$;

– the products of the non-derivative term $\left(-\frac{1}{2}\rho_l^2\right)$ in operators $\hat{D}(\rho_l; \gamma_{il})$ and contribution $\left(-\frac{1}{4}\frac{\rho_l^2}{\gamma_{il}}\right)$ in $J_{-1}^B(\rho_i, \gamma_{il})$, on which this operator acts – these give

$$+\frac{1}{8} \sum_{l \neq i} \varepsilon_{-1}^{[il]}(\gamma_{i\cdot}|\gamma_{l\cdot}) \frac{\rho_i^2\rho_l^2}{\gamma_{il}} = -\frac{1}{8} \sum_{l \neq i} \frac{\rho_i^2\rho_l^2\tilde{\Delta}_l}{\gamma_{il}^2}; \quad (77)$$

– the results of application of the double-derivative term $\left(-\frac{1}{16}\rho_l^4\partial^2/\partial\gamma_{il}^2\right)$ to $\left(-\tilde{\Delta}_l \cdot \log \gamma_{il}\right)$ – the ρ -independent contribution to $\varepsilon_{-1}^{[il]}(\gamma_{i\cdot}|\gamma_{l\cdot})J_{-1}^B(\rho_i, \gamma_{il}) = \frac{\tilde{\Delta}_l}{\gamma_{li}}J_{-1}^B(\rho_i, \gamma_{il})$ – these give

$$+\frac{1}{16} \sum_{l \neq i} \frac{\rho_l^4}{\gamma_{li}} \left(2\frac{\partial\tilde{\Delta}_l}{\partial\gamma_{li}} - \frac{\tilde{\Delta}_l}{\gamma_{li}}\right) = \frac{1}{8} \sum_{l \neq i} \frac{\rho_l^4\tilde{\Delta}_l}{\gamma_{li}} \left(\sum_{k \neq i, l} \frac{1}{\gamma_{lk}} + \frac{1}{2\gamma_{li}}\right) \quad (78)$$

All other terms are of different order in ρ^2 : the ones of lower order give rise to "universal" terms, forming *perturbative* prepotential, the ones of higher order, i.e. $O(\rho^6)$ contribute to higher non-perturbative corrections. Among the listed terms only (75) is of order $O(\rho^2)$. It gets combined with the logarithmic term $\frac{1}{4}\hat{S}_1 \log \rho_1^2$, and the two together are expressed through \hat{S}_1 , with the help of eq.(60).

3.3 Intermediate answer for the first non-perturbative correction

The answer for the integral $\Pi_i/2g_{n+1}$ is represented as a sum of two different kinds of terms: the "universal" ones (basically, the cut-off dependent and *perturbative* contributions),

$$\begin{aligned}
W_{n+1}(\Lambda) - \frac{1}{2} \sum_i \hat{S}_i \log \Lambda + \frac{1}{4} \hat{S}_i \log \left(\frac{1}{4} \hat{S}_i \right) - \frac{1}{4} \hat{S}_i - \\
-W_{n+1}(\gamma_i) - \frac{1}{4} \sum_{j \neq i} (\hat{S}_i - 2\hat{S}_j) \log \gamma_{ij}
\end{aligned} \tag{79}$$

and the *non-perturbative* ones, of which the first non-trivial is given by the sum of (76), (77) and (78):

$$\begin{aligned}
\frac{\rho_i^4 \tilde{\Delta}_i}{32} \sum_{\substack{j < k \\ j, k \neq i}} \frac{1}{\alpha_{ij} \alpha_{ik}} + \frac{1}{8} \sum_{j \neq i} \frac{\rho_i^2 \rho_j^2 (\tilde{\Delta}_i + \tilde{\Delta}_j)}{\gamma_{ij}^2} - \frac{1}{8} \sum_{j \neq i} \frac{\rho_i^2 \rho_j^2 \tilde{\Delta}_j}{\gamma_{ij}^2} + \\
+ \frac{1}{8} \sum_{j \neq i} \frac{\rho_j^4 \tilde{\Delta}_j}{\gamma_{ji}} \left(\frac{1}{2\gamma_{ji}} + \sum_{k \neq i, j} \frac{1}{\gamma_{jk}} \right)
\end{aligned} \tag{80}$$

(there is an obvious cancellation between the last term in the first line and a piece of the second term). Other contributions are of higher order $O(S^3)$ and can also be straightforwardly evaluated.

To transform this formula into a final answer, one still needs to get rid of remaining γ 's and express them through α 's and S 's.

4 Final formulas

In quadratic order in S all the γ 's in (80) can be just substituted by α 's, since the differences $\sigma_i = \gamma_i - \alpha_i = O(\rho^2) = O(S)$. In (79) the only γ -dependencies are in the second line: in the argument of $W_{n+1}(\gamma_i)$ (note that the *coefficients* of $W(x) = \prod (x - \alpha_i)$ are already expressed through α 's) and in $\log \gamma_{ij}$. Both things are easy to control: according to (110) in the quadratic order in S ,

$$\begin{aligned}
W_{n+1}(\gamma_i) &= W_{n+1}(\alpha_i) + \frac{1}{2} W_{n+1}''(\alpha_i) \sigma_i^2 + O(S^3) = \\
&= W_{n+1}(\alpha_i) + \frac{1}{8\Delta_i} \left(\sum_{j \neq i} \frac{\hat{S}_i - \hat{S}_j}{\alpha_{ij}} \right)^2 + O(S^3)
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
\log \gamma_{ij} &= \log \alpha_{ij} + \frac{\sigma_{ij}}{\alpha_{ij}} + O(S^2) = \log \alpha_{ij} - \\
&- \frac{1}{2\alpha_{ij}} \left[\frac{\hat{S}_i - \hat{S}_j}{\alpha_{ij}} \left(\frac{1}{\Delta_i} - \frac{1}{\Delta_j} \right) + \sum_{k \neq i, j} \left(\frac{\hat{S}_i - \hat{S}_k}{\alpha_{ik} \Delta_i} - \frac{\hat{S}_j - \hat{S}_k}{\alpha_{jk} \Delta_j} \right) \right]
\end{aligned} \tag{82}$$

The terms without σ 's, when substituted into (79) give rise to the "universal" terms, explicitly written in the expression (11) for the prepotential. The terms with σ 's add to (80) to give:

$$\begin{aligned}
& \frac{\hat{S}_i^2}{32\Delta_i} \sum_{\substack{j < k \\ j, k \neq i}} \frac{1}{\alpha_{ij}\alpha_{ik}} + \frac{1}{8} \sum_{j \neq i} \frac{\hat{S}_i \hat{S}_j}{\alpha_{ij}^2 \Delta_j} + \frac{1}{8} \sum_{j \neq i} \frac{\hat{S}_j^2}{\alpha_{ji} \Delta_j} \left(+ \frac{1}{2\alpha_{ji}} + \sum_{k \neq j, i} \frac{1}{\alpha_{jk}} \right) - \\
& - \frac{1}{8\Delta_i} \left(\sum_{j \neq i} \frac{\hat{S}_i - \hat{S}_j}{\alpha_{ij}} \right)^2 + \\
& + \frac{1}{8} \sum_{j \neq i} \frac{\hat{S}_i - 2\hat{S}_j}{\alpha_{ij}} \left[\frac{\hat{S}_i - \hat{S}_j}{\alpha_{ij}} \left(\frac{1}{\Delta_i} - \frac{1}{\Delta_j} \right) + \sum_{k \neq i, j} \left(\frac{\hat{S}_i - \hat{S}_k}{\alpha_{ik} \Delta_i} - \frac{\hat{S}_j - \hat{S}_k}{\alpha_{jk} \Delta_j} \right) \right]
\end{aligned} \tag{83}$$

Finally, the coefficients in the formula for $\Pi_i^{(3)} = (2\pi i)^{-1} \partial \mathcal{F}_3 / \partial S_i$,

$$\frac{1}{2g_{n+1}} \Pi_i^{(3)} = v_i(\alpha) \hat{S}_i^2 + \sum_{j \neq i} v_{i;j}(\alpha) \hat{S}_i \hat{S}_j + \sum_{j \neq i} v_{i;j}(\alpha) \hat{S}_j^2 + \sum_{\substack{j < k \\ j, k \neq i}} v_{i;jk}(\alpha) \hat{S}_j \hat{S}_k \tag{84}$$

are given by the following expressions:

$$\begin{aligned}
v_i &= \frac{1}{8} \left[\frac{1}{4\Delta_i} \sum_{\substack{j < k \\ j, k \neq i}} \frac{1}{\alpha_{ij}\alpha_{ik}} - \frac{1}{\Delta_i} \left(\sum_{j \neq i} \frac{1}{\alpha_{ij}} \right)^2 + \sum_{j \neq i} \frac{1}{\alpha_{ij}^2} \left(\frac{1}{\Delta_i} - \frac{1}{\Delta_j} \right) + \right. \\
& \quad \left. + \frac{1}{\Delta_i} \sum_{j \neq i} \left(\sum_{k \neq i, j} \frac{1}{\alpha_{ij}\alpha_{ik}} \right) \right] = \\
&= \frac{1}{8} \left[\frac{1}{4\Delta_i} \sum_{\substack{j < k \\ j, k \neq i}} \frac{1}{\alpha_{ij}\alpha_{ik}} - \sum_{j \neq i} \frac{1}{\alpha_{ij}^2 \Delta_j} \right]
\end{aligned} \tag{85}$$

$$\begin{aligned}
v_{i;j} &= \frac{1}{8} \left[\frac{1}{\alpha_{ij}^2 \Delta_j} + \frac{2}{\alpha_{ij} \Delta_i} \left(\frac{1}{\alpha_{ij}} + \sum_{k \neq i,j} \frac{1}{\alpha_{ik}} \right) - \frac{3}{\alpha_{ij}^2} \left(\frac{1}{\Delta_i} - \frac{1}{\Delta_j} \right) - \right. \\
&\quad \left. - \frac{2}{\alpha_{ij} \Delta_i} \sum_{k \neq i,j} \frac{1}{\alpha_{ik}} - \frac{1}{\alpha_{ij} \Delta_j} \sum_{k \neq i,j} \frac{1}{\alpha_{jk}} - \frac{1}{\alpha_{ij} \Delta_i} \sum_{k \neq i,j} \frac{1}{\alpha_{ik}} + \sum_{k \neq i,j} \frac{1}{\alpha_{ik} \alpha_{kj} \Delta_k} \right] = \\
&= \frac{1}{8} \left[-\frac{1}{\alpha_{ij}^2 \Delta_i} + \frac{4}{\alpha_{ij}^2 \Delta_j} - \frac{1}{\alpha_{ij} \Delta_i} \sum_{k \neq i,j} \frac{1}{\alpha_{ik}} - \frac{1}{\alpha_{ij} \Delta_j} \sum_{k \neq i,j} \frac{1}{\alpha_{jk}} - \sum_{k \neq i,j} \frac{1}{\alpha_{ik} \alpha_{jk} \Delta_k} \right] =
\end{aligned} \tag{86}$$

$$\begin{aligned}
&= \frac{1}{8} \left[-\frac{3}{\alpha_{ij}^2 \Delta_i} + \frac{2}{\alpha_{ij}^2 \Delta_j} - \frac{2}{\alpha_{ij} \Delta_i} \sum_{k \neq i,j} \frac{1}{\alpha_{ik}} \right] + \\
&+ \frac{1}{8} \left[\frac{2}{\alpha_{ij}^2 \Delta_i} + \frac{2}{\alpha_{ij}^2 \Delta_j} + \frac{1}{\alpha_{ij} \Delta_i} \sum_{k \neq i,j} \frac{1}{\alpha_{ik}} - \frac{1}{\alpha_{ij} \Delta_j} \sum_{k \neq i,j} \frac{1}{\alpha_{jk}} - \sum_{k \neq i,j} \frac{1}{\alpha_{ik} \alpha_{jk} \Delta_k} \right] = \\
&= \frac{1}{8} \left[-\frac{3}{\alpha_{ij}^2 \Delta_i} + \frac{2}{\alpha_{ij}^2 \Delta_j} - \frac{2}{\alpha_{ij} \Delta_j} \sum_{k \neq i,j} \frac{1}{\alpha_{ik}} \right] \\
v_{i;j} &= \frac{1}{8} \left[\frac{1}{2\alpha_{ij}^2 \Delta_j} - \frac{1}{\alpha_{ij} \Delta_j} \sum_{k \neq i,j} \frac{1}{\alpha_{jk}} - \frac{1}{\alpha_{ij}^2 \Delta_i} + \right. \\
&\quad \left. + \frac{2}{\alpha_{ij}^2} \left(\frac{1}{\Delta_i} - \frac{1}{\Delta_j} \right) + \frac{2}{\alpha_{ij} \Delta_j} \sum_{k \neq i,j} \frac{1}{\alpha_{jk}} \right] = \\
&= \frac{1}{8} \left[\frac{1}{\alpha_{ij}^2 \Delta_i} - \frac{3}{2} \frac{1}{\alpha_{ij}^2 \Delta_j} + \frac{1}{\alpha_{ij} \Delta_j} \sum_{k \neq i,j} \frac{1}{\alpha_{jk}} \right]
\end{aligned} \tag{87}$$

and

$$\begin{aligned}
v_{i;jk} &= \frac{1}{8} \left[-\frac{2}{\alpha_{ij} \alpha_{ik} \Delta_i} + \frac{2}{\alpha_{ij} \alpha_{ik} \Delta_i} - \frac{2}{\alpha_{ij} \alpha_{jk} \Delta_j} + \frac{2}{\alpha_{ik} \alpha_{ij} \Delta_i} - \frac{2}{\alpha_{ik} \alpha_{kj} \Delta_k} \right] = \\
&= \frac{1}{4} \left[\frac{1}{\alpha_{ij} \alpha_{ik} \Delta_i} + \frac{1}{\alpha_{ji} \alpha_{jk} \Delta_j} + \frac{1}{\alpha_{ki} \alpha_{kj} \Delta_k} \right]
\end{aligned} \tag{88}$$

Note, that integrability conditions for these relations are satisfied:

$$v_{i;jk} = v_{j;ik} = v_{k;ij} \tag{89}$$

for all triples of different i, j, k (of course, also $v_{i;jk} = v_{i;kj}$) and

$$v_{i;j} = 2v_{j;i} \quad (90)$$

for all $i \neq j$. The last equation follows from somewhat non-trivial relation,

$$\sum_{k \neq i,j} \frac{1}{\alpha_{ik}\alpha_{jk}\Delta_k} = \frac{2}{\alpha_{ij}^2\Delta_i} + \frac{2}{\alpha_{ij}^2\Delta_j} + \frac{1}{\alpha_{ij}} \sum_{k \neq i,j} \left(\frac{1}{\alpha_{ik}\Delta_i} - \frac{1}{\alpha_{jk}\Delta_j} \right) \quad (91)$$

It is just a triviality for $n = 2$, when only the first two terms survive and $\Delta_1 = -\Delta_2 = \alpha_{12}$, while already for $n = 3$ it already requires some calculation:

$$\frac{1}{\Delta_3^2} = \frac{2}{\alpha_{12}^2\Delta_1} + \frac{2}{\alpha_{12}^2\Delta_2} + \frac{1}{\Delta_1^2} + \frac{1}{\Delta_2^2} \quad (92)$$

For generic n it can be proved by showing that all the singularities at $\alpha_{kl} = 0$ cancel in the difference between the r.h.s. and l.h.s. (but poles up to the third order are present and analysis of residues is rather long).

From (84-88) it is easy to obtain the final expression (12) for the cubic term in the CIV-DV prepotential:

$$\begin{aligned} \frac{u_i}{i\pi g_{n+1}} &= \frac{(2\pi i) \cdot (2g_{n+1})}{(i\pi g_{n+1})^2} \frac{v_i}{3}, \quad i.e. \quad u_i = \frac{4}{3}v_i, \\ u_{i;j} &= \frac{4}{2}v_{i;j} = 4v_{j;i}, \\ u_{ijk} &= 4v_{i;jk} \end{aligned} \quad (93)$$

5 Conclusion

Now eq.(12) awaits an independent check, also from perturbative calculations for matrix models. A challenging problem is to further develop the machinery for evaluation of the higher-order terms in \mathcal{F}_k in the prepotential. Simplicity of the answer (12) strongly suggests that this should not be too difficult to do. A key point may be to understand the origins of identity (91) and its generalizations from the point of view of Seiberg-Witten theory or even via a free fermion method. It may provide a clue to the valuable tool to the entire problem.

Coming back to (12), one of its immediate applications is to experimental tests of the WDVV eqs [16], with [5, 6] and *without* additional T -moduli.

Obvious directions of the further analysis include:

- introduction of *flat* T -moduli instead of α_{ij} [5],
- interpretation of above calculation in terms of free fermions, matrix models and representation theory,
- application to the $N = 2$ SUSY prepotentials,
- comparison to instanton calculus *a la* refs.[14, 15, 4],
- generalization to numerous other models, of special interest being elliptic ones and non-exactly-solvable (non-eigenvalue) matrix models.

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A Relations between γ_i and α_i

These relations follow from identification of (3) and (6),

$$\prod_{L=1}^{2n} (x - \beta_L) \equiv \prod_{i=1}^n (x - \gamma_i^-)(x - \gamma_i^+) = \prod_{i=1}^n (x - \alpha_i)^2 + f_{n-1}(x), \quad (94)$$

where $\gamma_i^\pm = \gamma_i \pm \rho_i$. The entire polynomial can be represented as

$$\prod_{L=1}^{2n} (x - \beta_L) = \sum_{m=0}^{2n} (-)^m e_m^{(2n)} \{\beta\} x^{2n-m}, \quad (95)$$

and $f_{n-1}(x)$ does not contribute to the terms with $m \leq n$. This means that

$$e_m^{(2n)}(\{\gamma^-\}, \{\gamma^+\}) = e_m^{(2n)}(\{\alpha\}, \{\alpha\}), \quad \text{for } 1 \leq m \leq n. \quad (96)$$

At the same time, these symmetric polynomial of $2n$ variables can be decomposed into bilinear combinations of those of n variables,

$$e_m^{(2n)}(\{\gamma^-\}, \{\gamma^+\}) = \sum_{l=0}^m e_l^{(n)} \{\gamma^-\} e_{m-l}^{(n)} \{\gamma^+\} \quad (97)$$

and

$$e_m^{(2n)}(\{\alpha\}, \{\alpha\}) = \sum_{l=0}^m e_l^{(n)} \{\alpha\} e_{m-l}^{(n)} \{\alpha\}, \quad (98)$$

and we obtain the "sum rules":

$$\sum_{l=0}^m e_l^{(n)} \{\alpha\} e_{m-l}^{(n)} \{\alpha\} = \sum_{l=0}^m e_l^{(n)} \{\gamma^-\} e_{m-l}^{(n)} \{\gamma^+\} \quad \text{for } 1 \leq m \leq n. \quad (99)$$

For given n we have exactly n such equations, which are exactly what needed to express all γ_i through α 's and ρ 's. For higher $m > n$ the differences between the l.h.s. and the r.h.s. of (99) provide expressions for the coefficients of $f_{n-1}(x)$ through α 's and ρ 's.

The next step is to rewrite the sum rules (99) in terms of the more convenient basis of symmetric polynomial, $p_m\{\alpha\} = \sum_{i=1}^n \alpha_i^m$:

$$p_m\{\alpha\} = \frac{1}{2} (p_m\{\gamma^-\} + p_m\{\gamma^+\}) \quad \text{for } 1 \leq m \leq n. \quad (100)$$

In particular, in terms of matrices⁸ $\check{\alpha} = \text{diag}(\alpha_1, \dots, \alpha_n)$, $\check{\rho} = \text{diag}(\rho_1, \dots, \rho_n)$ and $\check{\gamma}^\pm = \text{diag}(\gamma_1^\pm, \dots, \gamma_n^\pm) = \check{\gamma} \pm \check{\rho}$ (note, that $[\check{\gamma}, \check{\rho}] = 0$),

⁸Transition from (99) to (100) is provided by the usual trick, especially simple in matrix notation: for

$$\begin{aligned} Z(\zeta; \check{\alpha}) &\equiv \sum_{l=0}^n \zeta^l (-)^l e_l^{(n)}(\{\alpha\}) = \prod_{l=0}^n (1 - \zeta \alpha_l) = \det_{n \times n} (I - \zeta \check{\alpha}) = \\ &= \exp \left(- \sum_{m=1}^{\infty} \frac{\zeta_m}{m} \text{Tr} \check{\alpha}^m \right) = \exp \left(- \sum_{m=1}^{\infty} \frac{\zeta_m}{m} p_m\{\alpha\} \right) \end{aligned}$$

$$\begin{aligned}
Tr \check{\alpha} &= Tr \check{\gamma}, \\
Tr \check{\alpha}^2 &= Tr \check{\gamma}^2 + Tr \check{\rho}^2, \\
Tr \check{\alpha}^3 &= Tr \check{\gamma}^3 + 3Tr \check{\gamma} \check{\rho}^2, \\
Tr \check{\alpha}^4 &= Tr \check{\gamma}^4 + 6Tr \check{\gamma}^2 \check{\rho}^2 + Tr \check{\rho}^4, \\
Tr \check{\alpha}^5 &= Tr \check{\gamma}^5 + 10Tr \check{\gamma}^3 \check{\rho}^2 + 5Tr \check{\gamma} \check{\rho}^4, \\
&\dots
\end{aligned} \tag{101}$$

For $n = 2$ the system of equations (101) reduces to a single quadratic equation, which can be solved exactly in all orders in ρ^2 :

$$\gamma_{12} = \sqrt{\alpha_{12}^2 - 2(\rho_1^2 + \rho_2^2)}. \tag{102}$$

For $n = 3$ the same can be done in terms of Cardano formulas for solutions of cubic equations.

For generic n one can get an expansion in powers of ρ 's, by iteratively solving the system of equations for $\sigma_i \equiv \gamma_i - \alpha_i$,

$$Tr \check{\alpha}^{m-1} \check{\sigma} = -\frac{m-1}{2} Tr \check{\alpha}^{m-2} \check{\rho}^2 + O(\rho^4) \tag{103}$$

($m = 1, \dots, n$). The determinant of the matrix appearing on the left-hand side of these equations is nothing but Van-der-Monde determinant $\Delta = \prod_{i < j}^n \alpha_{ij}$. Inverse of the Van-der-Monde matrix (α_i^{m-1}) is expressed through symmetric polynomials:

$$\sum_{m=1}^n (-)^{n-m-1} \frac{\Delta^{[i]}}{\Delta} e_{n-m}^{[i]}(\alpha) \cdot (\alpha_j^{m-1}) = \frac{\Delta^{[i]}}{\Delta} P_{[i]}(\alpha_j) = \delta_{ij} \frac{\Delta^{[i]} \Delta_i}{\Delta} = \delta_{ij} \tag{104}$$

Here

$$P^{[i]}(x) \equiv \prod_{k \neq i} (x - \alpha_k) = \frac{P(x)}{(x - \alpha_i)} \tag{105}$$

and

$$\Delta^{[i]} = \prod_{\substack{j < k \\ j, k \neq i}} \alpha_{jk} = \Delta \prod_{k \neq i} \frac{1}{\alpha_{ik}}, \quad \Delta_i = \prod_{k \neq i} \alpha_{ik} \tag{106}$$

Therefore

$$\begin{aligned}
\sigma_i &= -\frac{1}{2} \frac{\Delta^{[i]}}{\Delta} \sum_{m=1}^n (-)^{n-m-1} e_{n-m}^{[i]}(\alpha) \cdot (m-1) Tr \check{\alpha}^{m-2} \check{\rho}^2 = \\
&= -\frac{1}{2 \Delta_i} \sum_{j=1}^n P^{[i]'}(\alpha_j) \rho_j^2 = -\frac{1}{2} \prod_{k \neq i} \frac{1}{\alpha_{ik}} \left(P^{[i]'}(\alpha_i) \rho_i^2 + \sum_{j \neq i} P^{[i]'}(\alpha_j) \rho_j^2 \right)
\end{aligned} \tag{107}$$

the sum rules (99) imply that

$$Z(\zeta; \check{\alpha})^2 = Z(\zeta; \check{\gamma}^-) Z(\zeta; \check{\gamma}^+),$$

for the first n terms of expansion in powers of ζ . This, obviously, implies (100).

so that finally

$$\sigma_i = -\frac{1}{2} \left(\rho_i^2 \sum_{k \neq i} \frac{1}{\alpha_{ik}} + \sum_{j \neq i} \rho_j^2 \frac{\prod_{k \neq i, j} \alpha_{jk}}{\prod_{k \neq i} \alpha_{ik}} \right) + O(\rho^4) \quad (108)$$

If we now use (59) to express

$$\rho_i^2 = \hat{S}_i / \Delta_i + O(S^2), \quad \Delta_i = \prod_{j \neq i} \alpha_{ij}, \quad (109)$$

eq.(108) acquires the simple final form:

$$\sigma_i = -\frac{1}{2\Delta_i} \sum_{j \neq i} \frac{S_i - S_j}{\alpha_{ij}} + O(S^2). \quad (110)$$

B Examples of $n = 2$ and $n = 3$

For illustrative purposes we present here a more detailed calculations for the simplest cases of $n = 2$ and $n = 3$.

B.1 $n = 2$:

$$\begin{aligned} -\frac{1}{2g_3} S_1 &= A_0^{[1]}(\rho_1; \gamma_{12}) + \hat{D}(\rho_2; \gamma_{12}) A_1^{[1]}(\rho_1; \gamma_{12}) = \\ &= J_1^A(\rho_1) + \gamma_{12} J_0^A(\rho_1) + \hat{D}(\rho_2; \gamma_{12}) J_{-1}^A(\rho_1, \gamma_{12}) = \\ &= i\pi \left(c_1 \rho_1^2 \gamma_{12} + \sum_{k=1}^{\infty} c_k \rho_1^{2k} \hat{D}(\rho_2; \gamma_{12}) \frac{1}{\gamma_{12}^{2k-1}} \right) = \\ &= -\frac{i\pi}{2} \left(\rho_1^2 \gamma_{12} - \sum_{k,l=0}^{\infty} \frac{\rho_1^{2k+2} \rho_2^{2l+2}}{2^{2k+2l+1}} \frac{(2k+2l)!}{k!(k+1)!l!(l+1)!} \frac{1}{\gamma_{12}^{2k+2l+1}} \right) \end{aligned} \quad (111)$$

Similarly,

$$\begin{aligned} -\frac{1}{2g_3} S_2 &= A_0^{[2]}(\rho_2; \gamma_{21}) + \hat{D}(\rho_1; \gamma_{21}) A_1^{[2]}(\rho_2; \gamma_{21}) = \\ &= J_1^A(\rho_2) + \gamma_{21} J_0^A(\rho_2) + \hat{D}(\rho_1; \gamma_{21}) J_{-1}^A(\rho_2, \gamma_{21}) = \\ &= i\pi \left(c_1 \rho_2^2 \gamma_{21} + \sum_{l=1}^{\infty} c_l \rho_2^{2l} \hat{D}(\rho_1; \gamma_{21}) \frac{1}{\gamma_{21}^{2l-1}} \right) = \\ &= -\frac{i\pi}{2} \left(\rho_1^2 \gamma_{21} - \sum_{k,l=0}^{\infty} \frac{\rho_1^{2k+2} \rho_2^{2l+2}}{2^{2k+2l+1}} \frac{(2k+2l)!}{k!(k+1)!l!(l+1)!} \frac{1}{\gamma_{21}^{2k+2l+1}} \right) = \\ &= \frac{1}{2g_3} S_1 - \frac{i\pi}{2} (\rho_1^2 - \rho_2^2) \gamma_{12}, \end{aligned} \quad (112)$$

so that

$$S_1 + S_2 = i\pi g_3(\rho_1^2 - \rho_2^2)\gamma_{12} \quad (113)$$

We do not write here an even more sophisticated general expressions for Π_1 and Π_2 : they can be easily obtained in the form of *triple* series in powers of ρ^2 , but after that one still needs to express ρ_1^2 and ρ_2^2 through S_1 and S_2 from (111) and (112) and substitute into the series for Π_1 and Π_2 . However, the problem of inverting (111) and (112) is not resolved yet.

Instead, below we concentrate on the first terms of these expansions.

$$\begin{aligned} \hat{S}_1 &= \frac{S_1}{i\pi g_3} = \rho_1^2 \gamma_{12} - \frac{1}{2} \frac{\rho_1^2 \rho_2^2}{\gamma_{12}} + O(\rho^6), \\ \hat{S}_2 &= \frac{S_2}{i\pi g_3} = -\rho_2^2 \gamma_{12} + \frac{1}{2} \frac{\rho_1^2 \rho_2^2}{\gamma_{12}} + O(\rho^6), \end{aligned} \quad (114)$$

so that

$$\begin{aligned} \rho_1^2 &= \frac{\hat{S}_1}{\gamma_{12}} - \frac{\hat{S}_1 \hat{S}_2}{2\gamma_{12}^4} + O(\rho^6) \\ \rho_2^2 &= -\frac{\hat{S}_2}{\gamma_{12}} - \frac{\hat{S}_1 \hat{S}_2}{2\gamma_{12}^4} + O(\rho^6) \end{aligned} \quad (115)$$

The Π period is (remember that $D(\rho; \gamma_{ik})$ commutes with $\Lambda_i!$):

$$\begin{aligned}
\frac{\Pi}{2g_3} &= J_1^B(\rho_1) + \gamma_{12} J_0^B(\rho_1) - \frac{1}{2} \left(\rho_2^2 + \frac{\rho_2^4}{8} \frac{\partial^2}{\partial \gamma_{12}^2} \right) J_{-1}^B(\rho_1, \Lambda_1) = \\
&= \left(\frac{1}{24} Z_1^3 - \frac{1}{8} \rho_1^2 Z_1 \right) + \left(\frac{1}{2} \gamma_{12} \rho_1^2 \log \frac{\rho_1}{2\Lambda} + \frac{1}{8} \gamma_{12} Z_1^2 \right) \\
&- \frac{1}{2} \left(\rho_2^2 + \frac{\rho_2^4}{8} \frac{\partial^2}{\partial \gamma_{12}^2} \right) \left(\Lambda_1 + \gamma_{12} \log \frac{\gamma_{12}}{\Lambda} + \frac{\rho_1^2}{2\gamma_{12}} \log \frac{\rho_1}{2\gamma_{12}} - \frac{\rho_1^2}{4\gamma_{12}} \right) + O(\rho^6) = \\
&= \frac{1}{3} (\Lambda^3 - 3\Lambda^2 \gamma_1 + 3\Lambda \gamma_1^2 - \gamma_1^3) + \frac{1}{2} \gamma_{12} (\Lambda^2 - 2\Lambda \gamma_1 + \gamma_1^2) - \frac{1}{2} (\rho_1^2 + \rho_2^2) (\Lambda - \gamma_1) - \frac{1}{4} \gamma_{12} \rho_1^2 - \\
&- \frac{1}{2} \gamma_{12} (\rho_1^2 - \rho_2^2) \log \Lambda + \frac{1}{2} \gamma_{12} \rho_1^2 \log \frac{\rho_1}{2} - \frac{1}{2} \gamma_{12} \rho_2^2 \log \gamma_{12} - \frac{1}{4} \frac{\rho_1^2 \rho_2^2}{\gamma_{12}} \log \frac{\rho_1}{2\gamma_{12}} - \\
&- \frac{\rho_2^4}{16\gamma_{12}} + \frac{\rho_1^2 \rho_2^2}{8\gamma_{12}} + \\
&+ O(\rho^6) =
\end{aligned} \tag{116}$$

$$\begin{aligned}
&= \frac{1}{3} (\Lambda^3 - \gamma_1^3) - \frac{1}{2} (\gamma_1 + \gamma_2) (\Lambda^2 - \gamma_1^2) + \left(\gamma_1 \gamma_2 - \frac{1}{2} (\rho_1^2 + \rho_2^2) \right) (\Lambda - \gamma_1) - \\
&- \frac{1}{2} \gamma_{12} (\rho_1^2 - \rho_2^2) \log \Lambda + \frac{1}{4} \left(\gamma_{12} \rho_1^2 - \frac{\rho_1^2 \rho_2^2}{2\gamma_{12}} \right) \log \frac{\rho_1^2}{4} - \frac{1}{2} \left(\gamma_{12} \rho_2^2 - \frac{\rho_1^2 \rho_2^2}{2\gamma_{12}} \right) \log \gamma_{12} - \\
&- \frac{1}{4} \gamma_{12} \left(\rho_1^2 - \frac{\rho_1^2 \rho_2^2}{2\gamma_{12}} \right) - \frac{\rho_2^4}{16\gamma_{12}} + \\
&+ O(\rho^6) =
\end{aligned}$$

$$\begin{aligned}
&= W_3(\Lambda) - W_3(\gamma_1) - \frac{1}{2} (\hat{S}_1 + \hat{S}_2) \log \Lambda - \frac{1}{4} \hat{S}_1 + \\
&+ \frac{1}{4} \hat{S}_1 \log \frac{\hat{S}_1}{4} + \left(\frac{1}{2} \hat{S}_2 - \frac{1}{4} \hat{S}_1 \right) \log \gamma_{12} - \frac{\hat{S}_1 \hat{S}_2}{8\gamma_{12}^3} - \frac{S_2^2}{16\gamma_{12}^3} + O(S^3)
\end{aligned}$$

At the last stage we substituted the expression for ρ_1^2 through \hat{S}_1 and \hat{S}_2 in the argument of the logarithm,

$$\frac{1}{4} \hat{S}_1 \log \frac{\rho_1^2}{4} = \frac{1}{4} \hat{S}_1 \left(\log \frac{\hat{S}_1}{4} - \log \gamma_{12} - \frac{\hat{S}_2}{2\gamma_{12}^3} \right) \tag{117}$$

and moved the two newly emerging terms closer to the similar ones, coming from other sources. Finally,

$$\begin{aligned}
\frac{\Pi}{2g_3} &= W_3(\Lambda) - \frac{1}{2} (\hat{S}_1 + \hat{S}_2) \log \Lambda + \frac{1}{4} \hat{S}_1 \log \frac{\hat{S}_1}{4} - \frac{1}{4} \hat{S}_1 - \\
&- W_3(\gamma_1) - \frac{1}{4} (\hat{S}_1 - 2\hat{S}_2) \log \gamma_{12} - \frac{\hat{S}_1 \hat{S}_2}{8\gamma_{12}^3} - \frac{S_2^2}{16\gamma_{12}^3} + O(S^3)
\end{aligned} \tag{118}$$

Now, it remains to substitute

$$\begin{aligned}
W_3(\gamma_1) &= W_3(\alpha_1) + \frac{1}{2}\alpha_{12}\sigma^2, \\
\sigma = \gamma_1 - \alpha_1 &= -(\gamma_2 - \alpha_2) = \frac{1}{2}(\gamma_{12} - \alpha_{12}) = -\frac{\rho_1^2 + \rho_2^2}{2\alpha_{12}} + O(\rho^2) = \\
&= -\frac{\hat{S}_1 - \hat{S}_2}{2\alpha_{12}^3} + O(S^2)
\end{aligned} \tag{119}$$

and

$$\log \gamma_{12} = \log \alpha_{12} - \frac{\rho_1^2 + \rho_2^2}{\alpha_{12}} + O(\rho^2) = \log \alpha_{12} - \frac{\hat{S}_1 - \hat{S}_2}{\alpha_{12}^3} + O(S^2) \tag{120}$$

to obtain:

$$\begin{aligned}
\frac{\Pi}{2g_3} &= W_3(\Lambda) - W_3(\alpha_1) - \frac{1}{2}(\hat{S}_1 + \hat{S}_2) \log \Lambda + \frac{1}{4}\hat{S}_1 \log \frac{\hat{S}_1}{4} - \frac{1}{4}\hat{S}_1 + \\
&\quad + \frac{1}{8\alpha_{12}^3} \left(\hat{S}_1^2 - 5\hat{S}_1\hat{S}_2 + \frac{5}{2}\hat{S}_2^2 \right)
\end{aligned} \tag{121}$$

(from $2(S_1 - S_2)(S_1 - 2S_2) - (S_1 - S_2)^2 - S_1S_2 - \frac{1}{2}S_2^2$).

B.2 n = 3 :

The starting expression is

$$\begin{aligned}
\int_1 dS_{DV} &= J_2(\rho_1; 0) + (\gamma_{12} + \gamma_{13})J_1(\rho_1; 0) + \gamma_{12}\gamma_{13}J_0(\rho_1; 0) + \\
&\quad + \hat{D}(\rho_2, \gamma_{12}) [J_0(\rho_1; 0) + \gamma_{23}J_{-1}(\rho_1; \gamma_{12}|\Lambda_1)] + \\
&\quad + \hat{D}(\rho_3, \gamma_{13}) [J_0(\rho_1; 0) - \gamma_{23}J_{-1}(\rho_1; \gamma_{13}|\Lambda_1)] + \\
&\quad + \hat{D}(\rho_2, \gamma_{12})\hat{D}(\rho_3, \gamma_{13})\frac{1}{\gamma_{23}} [J_{-1}(\rho_1; \gamma_{12}|\Lambda_1) - J_{-1}(\rho_1; \gamma_{13}|\Lambda_1)] + O(\rho^6)
\end{aligned} \tag{122}$$

Expressions for S -integrals:

$$\begin{aligned}
-\frac{S_1}{2\pi i g_4} &= -\frac{1}{8}\rho_1^4 - \frac{1}{2}\gamma_{12}\gamma_{13}\rho_1^2 - \\
&\quad - \frac{1}{2} \left(\rho_2^2 + \frac{\rho_2^4}{8} \frac{\partial^2}{\partial \gamma_{12}^2} \right) \left(-\frac{1}{2}\rho_1^2 - \gamma_{23} \frac{\rho_1^2}{2\gamma_{12}} \right) - \\
&\quad - \frac{1}{2} \left(\rho_3^2 + \frac{\rho_3^4}{8} \frac{\partial^2}{\partial \gamma_{13}^2} \right) \left(-\frac{1}{2}\rho_1^2 + \gamma_{23} \frac{\rho_1^2}{2\gamma_{13}} \right) + \\
&\quad + \frac{1}{4}\rho_2^2\rho_3^2\frac{1}{\gamma_{23}} \left(-\frac{\rho_1^2}{2\gamma_{12}} + \frac{\rho_1^2}{2\gamma_{13}} \right) + O(\rho^6)
\end{aligned} \tag{123}$$

In fact, not all of these terms actually contribute to the S -integrals in the order $O(\rho^4)$ (however, their counterparts in the expressions for Π -integrals *do* contribute). In result,

$$\begin{aligned}\hat{S}_1 &= \frac{S_1}{i\pi g_4} = \gamma_{12}\gamma_{13}\rho_1^2 + \frac{1}{4}\rho_1^4 - \frac{\gamma_{13}}{2\gamma_{12}}\rho_1^2\rho_2^2 - \frac{\gamma_{12}}{2\gamma_{13}}\rho_1^2\rho_3^2 + O(\rho^6) = \\ &= \gamma_{12}\gamma_{13}\rho_1^2 \left(1 + \frac{\rho_1^2}{4\gamma_{12}\gamma_{13}} - \frac{\rho_2^2}{2\gamma_{12}^2} - \frac{\rho_3^2}{2\gamma_{13}^2} \right) + O(\rho^6)\end{aligned}\tag{124}$$

Similarly, for \hat{S}_2 and \hat{S}_3 .

Expressions for ρ 's through $\hat{S}_i = S_i/i\pi g_4$:

$$\begin{aligned}\rho_1^2 &= \frac{\hat{S}_1}{\gamma_{12}\gamma_{13}} \left(1 - \frac{1}{4} \frac{\hat{S}_1}{\gamma_{12}^2\gamma_{13}^2} + \frac{1}{2} \frac{\hat{S}_2}{\gamma_{21}^2\gamma_{23}} + \frac{1}{2} \frac{\hat{S}_3}{\gamma_{31}^2\gamma_{32}} \right) + O(S^3); \\ \rho_2^2 &= \frac{\hat{S}_2}{\gamma_{21}\gamma_{23}} \left(1 + \frac{1}{2} \frac{\hat{S}_1}{\gamma_{12}^2\gamma_{13}} - \frac{1}{4} \frac{\hat{S}_2}{\gamma_{21}^2\gamma_{23}^2} + \frac{1}{2} \frac{\hat{S}_3}{\gamma_{23}^2\gamma_{13}} \right) + O(S^3); \\ \rho_3^2 &= \frac{\hat{S}_3}{\gamma_{31}\gamma_{32}} \left(1 + \frac{1}{2} \frac{\hat{S}_1}{\gamma_{13}^2\gamma_{12}} + \frac{1}{2} \frac{\hat{S}_2}{\gamma_{23}^2\gamma_{21}} - \frac{1}{4} \frac{\hat{S}_3}{\gamma_{13}^2\gamma_{23}^2} \right) + O(S^3).\end{aligned}\tag{125}$$

Now, the Π integral:

$$\begin{aligned}\frac{\Pi_1}{2g_4} &= \frac{1}{64}Z_1^4(\rho_1) + \frac{1}{8}\rho_1^4 \log \frac{\rho_1}{2\Lambda} + \\ &+ (\gamma_{12} + \gamma_{13}) \left(\frac{1}{24}Z_1^3(\rho_1) - \frac{1}{8}Z_1(\rho_1) \right) + \\ &\gamma_{12}\gamma_{13} \left(\frac{1}{8}Z_1^2(\rho_1) + \frac{1}{2}\rho_1^2 \log \frac{\rho_1}{2\Lambda} \right) - \\ &- \frac{1}{2} \left(\rho_2^2 + \frac{\rho_2^4}{8} \frac{\partial^2}{\partial \gamma_{12}^2} \right) \left[\left(\frac{\rho_1^2}{2} \log \frac{\rho_1}{2\Lambda} + \frac{1}{8}Z_1^2(\rho_1) \right) + \right. \\ &+ \gamma_{23} \left(\Lambda_1 + \gamma_{12} \log \frac{\gamma_{12}}{\Lambda} + \frac{\rho_1^2}{2\gamma_{12}} \log \frac{\rho_1}{2\gamma_{12}} - \frac{\rho_1^2}{4\gamma_{12}} \right) \Big] + \\ &- \frac{1}{2} \left(\rho_3^2 + \frac{\rho_3^4}{8} \frac{\partial^2}{\partial \gamma_{13}^2} \right) \left[\left(\frac{\rho_1^2}{2} \log \frac{\rho_1}{2\Lambda} + \frac{1}{8}Z_1^2(\rho_1) \right) - \right. \\ &- \gamma_{23} \left(\Lambda_1 + \gamma_{13} \log \frac{\gamma_{13}}{\Lambda} + \frac{\rho_1^2}{2\gamma_{13}} \log \frac{\rho_1}{2\gamma_{13}} - \frac{\rho_1^2}{4\gamma_{13}} \right) \Big] + \\ &+ \frac{\rho_2^2\rho_3^2}{4\gamma_{23}} \left(\gamma_{12} \log \frac{\gamma_{12}}{\Lambda} - \gamma_{13} \log \frac{\gamma_{13}}{\Lambda} \right) + O(\rho^6)\end{aligned}\tag{126}$$

where

$$\begin{aligned}Z_1^4 &= 16\Lambda_1^4 - 16\Lambda_1^2\rho_1^2 + 2\rho_1^4 + O(1/\Lambda), \\ Z_1^3 &= 8\Lambda_1^3 - 6\Lambda_1\rho_1 + O(1/\Lambda), \\ Z_1^2 &= 4\Lambda_1^2 - 2\rho_1^2 + O(1/\Lambda), \\ Z_1 &= 2\Lambda_1 + O(1/\Lambda).\end{aligned}\tag{127}$$

Long transformation finally provides:⁹

$$\begin{aligned}
\frac{\Pi_1}{2g_4} = & W_4(\Lambda) - \frac{1}{2}(\hat{S}_1 + \hat{S}_2 + \hat{S}_3) \log \Lambda + \frac{\hat{S}_1}{4} \log \frac{\hat{S}_1}{4} - \frac{\hat{S}_1}{4} - \\
& - W_4(\gamma_1) - \frac{1}{4}(\hat{S}_1 - 2\hat{S}_2) \log \gamma_{12} - \frac{1}{4}(\hat{S}_1 - 2\hat{S}_3) \log \gamma_{13} + \\
& + \frac{1}{32} \frac{\hat{S}_1^2}{\gamma_{12}^2 \gamma_{13}^2} + \frac{\hat{S}_2^2}{16} \left(\frac{2}{\gamma_{12}^2 \gamma_{23}^2} - \frac{1}{\gamma_{12}^3 \gamma_{23}} \right) + \frac{\hat{S}_3^2}{16} \left(\frac{2}{\gamma_{13}^2 \gamma_{23}^2} + \frac{1}{\gamma_{13}^3 \gamma_{23}} \right) - \frac{1}{8} \frac{\hat{S}_1 \hat{S}_2}{\gamma_{12}^3 \gamma_{23}} - \frac{1}{8} \frac{\hat{S}_1 \hat{S}_3}{\gamma_{13}^3 \gamma_{23}} \\
& + O(S^3, \Lambda^{-1})
\end{aligned} \tag{128}$$

In this expression the coefficients of $W'_4(x) = \prod_{i=1}^n (x - \alpha_i)$ are already expressed through α . It remains to express γ 's through α 's in the *argument* of $W(\gamma_1)$ and in logarithmic terms – all in the second line. In the given order in S , the γ 's in the third line can be simply substituted by α 's. This procedure literally repeats the generic calculation in s.4 and we do not repeat it here.

⁹Note that there are *no more* corrections to the cut-off- Λ -dependent terms, namely, no further contributions with higher powers of ρ 's. Still, Λ and γ_i (isolated γ_i not in the form of difference γ_{ij}) enter only as arguments of $W_4(x)$ with coefficients made out of α 's, not γ 's! This is made possible by *exact* sum rules (100), which do not get any higher-order corrections.

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